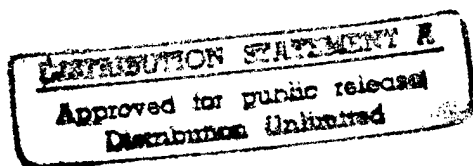


VOLUME II  
FLYING QUALITIES PHASE

CHAPTER 4  
EQUATIONS OF MOTION



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#### 4.1 INTRODUCTION

The performance of an aircraft can adequately be described by assuming the aircraft is a point mass concentrated at the aircraft's center of gravity (cg). The flying qualities of an aircraft, on the other hand, cannot be described in such a simple manner. The flying qualities of an aircraft must, instead, be described analytically as motions of the aircraft's cg as well as motions of the airframe about the cg, both of which are caused by aerodynamic, thrust and other forces and moments. In addition, the aircraft must be considered a three dimensional body and not a point mass.

The applied forces and moments on the aircraft and the resulting response of the aircraft are traditionally described by a set of equations known as the aircraft equations of motion. This chapter presents the form of the aircraft equations of motion used in the Flying Qualities phase of the USAF Test Pilot School curriculum.

The theory presented in this chapter incorporates certain simplifying assumptions to make the main elements of the subject clearer. The equations that will be developed are not as rigorous (and complicated) as those used for design of modern aircraft, but the basic method is valid and will provide analysis techniques that are accurate enough to gain an insight into aircraft flying qualities. With the aid of high speed computers the aircraft designers' more rigorous theoretical calculations, modified by data obtained from the wind tunnel, can often give results which closely predict aircraft flying qualities. This is of substantial benefit in the development cycle of new aircraft.

#### 4.2 OVERVIEW

An aircraft has six degrees of freedom (if it is assumed to be rigid), which means it has six paths it is free to follow: it can move forward, sideways, and down; and it can rotate about its axes with yaw, pitch, and roll. In order to describe the state of a system that has six degrees of freedom, values for six variables (unknowns) are necessary. To solve for these six unknowns, six simultaneous equations are necessary. For an aircraft, these are known as the aircraft equations of motion.

The full aircraft equations of motion (given in sections 4.10 and 4.11)

reflect a rather complicated relationship between the forces and moments on the aircraft, and the resulting aircraft motion. The derivation of the equations, however, follows a very simple pattern starting from Newton's second law for translational and rotational motions.

Newton's second law for translational motions is

$$\bar{F} = \frac{d}{dt} (m \bar{V}) \quad (4.1)$$

where  $\bar{F}$  is the sum of the externally applied forces and  $m\bar{V}$  is linear momentum.

Newton's second law for rotational motions is

$$\bar{G} = \frac{d}{dt} (\bar{H}) \quad (4.2)$$

where  $\bar{G}$  is the sum of the externally applied moments and  $\bar{H}$  is angular momentum.

$\bar{F}$  and  $\bar{G}$  are both vector quantities which can each be represented by three component equations (corresponding to three dimensional space). The translational equation, therefore, describes the aircraft with respect to its three translational degrees of freedom, while the rotational equation describes the aircraft with respect to its three rotational degrees of freedom. Newton's second law, therefore, yields six equations for the six degrees of freedom of a rigid body.

In order to derive the equations of motion, each side of Newton's equations are expanded to yield the following six nonlinear differential equations:

$$\text{Longitudinal} \left\{ \begin{array}{l} F_x = m (\dot{U} + QW - RV) \\ F_z = m (\dot{W} + PV - QU) \\ G_y = \dot{Q} I_y - PR (I_z - I_x) + (P^2 - R^2) I_{xz} \end{array} \right. \quad \begin{array}{l} (4.3) \\ (4.4) \\ (4.5) \end{array}$$

$$\text{Lateral-Directional} \left\{ \begin{array}{l} F_y = m (\dot{V} + RU - PW) \\ G_x = \dot{P} I_x + QR (I_z - I_y) - (\dot{R} + PQ) I_{xz} \\ G_z = \dot{R} I_z + PQ (I_y - I_x) + (QR - \dot{P}) I_{xz} \end{array} \right. \quad \begin{array}{l} (4.6) \\ (4.7) \\ (4.8) \end{array}$$

The Left-Hand Side (LHS) of these equations represent the applied forces and moments on the aircraft while the Right-Hand Side (RHS) stands for the aircraft's response to these forces and moments. Small perturbation theory will be used to linearize these equations so they can be solved. This will also yield terms known as stability derivatives which indicate the influence of various aircraft characteristics on the resulting aircraft motions, and are useful in comparing aircraft, calculating MIL-SPEC requirements, etc. The equations will also be used to derive aircraft transfer functions which will be a fundamental part of the mathematical modeling of the aircraft and its control system in later chapters.

Prior to beginning the derivation of the aircraft equations of motion, a discussion will be presented of aircraft sign conventions and coordinate systems. There will also be many abbreviations and symbols used during the derivation of the equations of motion and in subsequent flying qualities chapters. A list of abbreviations and symbols is given in the last subsection of this chapter.

#### 4.3 SIGN CONVENTIONS

The sign convention used at the Test Pilot School and the Flight Test Center defines a positive control movement or deflection as one that causes a positive aircraft movement (right yaw, pitch up, right roll). Figure 4.1 shows this sign convention and gives the positive directions for many of the variables that appear in the equations of motion. The TPS (AFFTC) sign convention does not conform to the convention used by NASA and some reference textbooks. Sign conventions will be discussed in greater detail in the flight control systems chapter.

#### 4.4 COORDINATE SYSTEMS

There are many coordinate systems that are useful in the analysis of vehicle motion. We will be concerned with three of these coordinate systems: inertial, earth axis, and vehicle axis. According to convention, all coordinate systems used will be right-hand orthogonal.

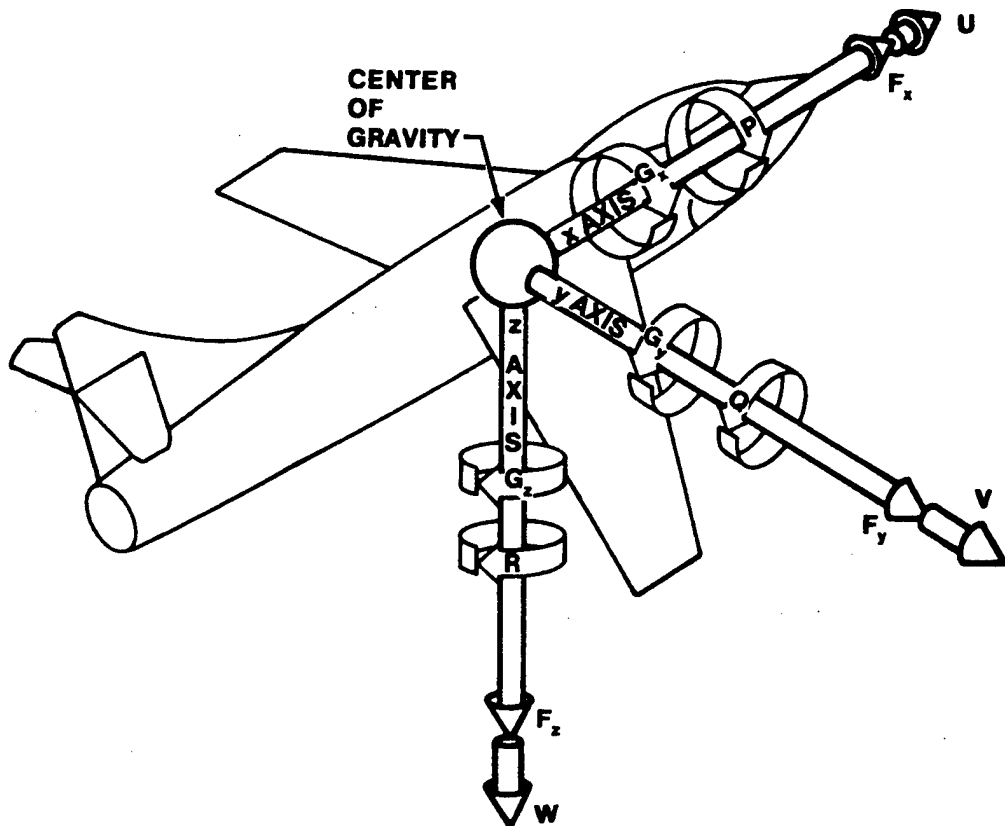


FIGURE 4.1. VEHICLE FIXED AXIS SYSTEM AND NOTATION

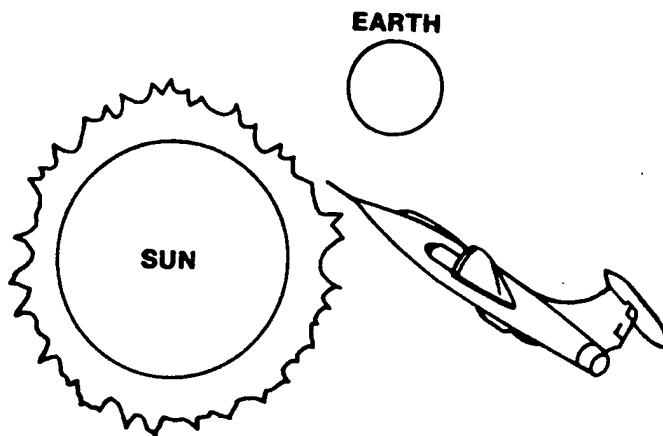


FIGURE 4.2. THE INERTIAL COORDINATE SYSTEM

#### 4.4.1 Inertial Coordinate System

An inertial coordinate system is defined as a system in which Newton's second law is valid. The equations of motion must, therefore, be determined in an inertial coordinate system. Another way of defining the inertial coordinate system is to assume it is an axis system fixed in space that has no relative motion (Figure 4.2).

Experience with physical observations can be used to determine whether a particular reference system can properly be assumed to be an inertial coordinate system for the application of Newton's laws to a particular problem. For space dynamics in our solar system, the sun axis system is a sufficient approximation for an inertial system. For aircraft, the earth axis system is usually a sufficient approximation for an inertial coordinate system.

#### 4.4.2 Earth Axis System

There are two earth axis systems, the fixed and the moving. Both will be referred to with the letters XYZ for the three coordinate axes. An example of a moving earth axis system is an inertial navigation platform. An example of a fixed earth axis system is a radar site (Figure 4.3).

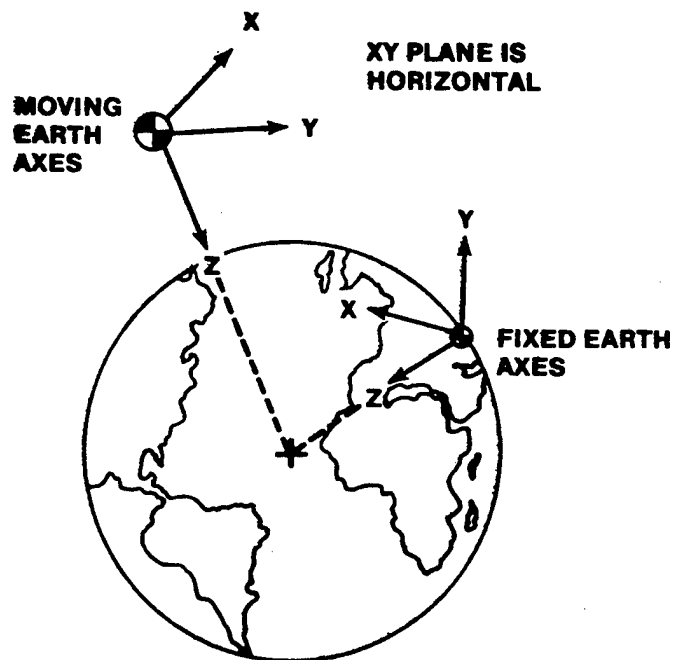


FIGURE 4.3. THE EARTH AXIS SYSTEMS

In both earth axis systems, the Z-axis points toward the center of the earth along the gravitational vector,  $\bar{g}$ . The XY-plane is parallel to the local horizontal while the orientation of the X-axis is arbitrarily defined (often defined as North). The two earth axis systems are distinguished by the location of their origins. The origin of the fixed system is usually taken as an arbitrary location on the earth's surface. The origin of the moving system is usually taken as the vehicle's cg. What distinguishes the moving earth axis system from the vehicle axis system discussed in the next subsection is that the moving earth axes are not fixed in orientation with respect to the vehicle. They are instead fixed with respect to local vertical. In the rest of this chapter, the XYZ (upper case) system will be assumed to be the fixed earth axis system unless otherwise noted.

#### 4.4.3 Vehicle Axis Systems

These coordinate systems have origins fixed to the vehicle, and axes defined with respect to the vehicle. There are many different types, four of which are commonly used for describing aircraft motion: the body axis system, the stability axis system, the principal axis system, and the wind axis system. The body and the stability axis systems are the only two that will be used during this course.

4.4.3.1 Body Axis System. The body axis system (Figure 4.4) is the most general kind of axis system in which the origin and axes are fixed to a rigid body. The use of axes fixed to the vehicle ensures that the moments and products of inertia in the equations of motion are constant, to the extent that mass can also be considered constant, and that aerodynamic forces and moments depend only upon the relative velocity orientation angles  $\alpha$  and  $\beta$ . The body fixed axis system is also the natural frame of reference for most vehicle-borne observations and measurements of the vehicle's motion and will be referred to as the xyz (lower case) system.

In the body axis system the unit vectors are  $\bar{i}$ ,  $\bar{j}$  and  $\bar{k}$  with origins at the vehicle cg. The positive x-axis points forward along a vehicle horizontal reference line with the positive y-axis out the right wing. The positive z-axis points downward out the bottom of the vehicle, usually such that the xz-plane is the vehicle plane of symmetry.

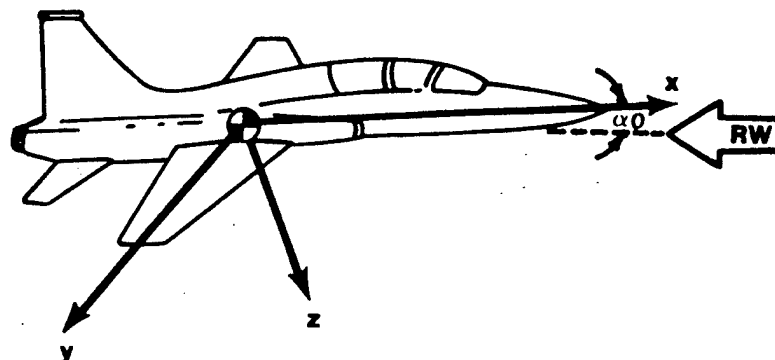


FIGURE 4.4. BODY AXIS SYSTEM

4.4.3.2 Stability Axis System. Stability axes are specialized body axes (Figure 4.5) in which the orientation of the vehicle axis system is determined by the equilibrium flight condition. The  $x_s$ -axis is selected to be coincident with the relative wind at the start of the motion. This initial alignment does not alter the body-fixed nature of the axis system; however, the alignment of the axis system with respect to the body changes as a function of the equilibrium condition. If the reference flight condition is not

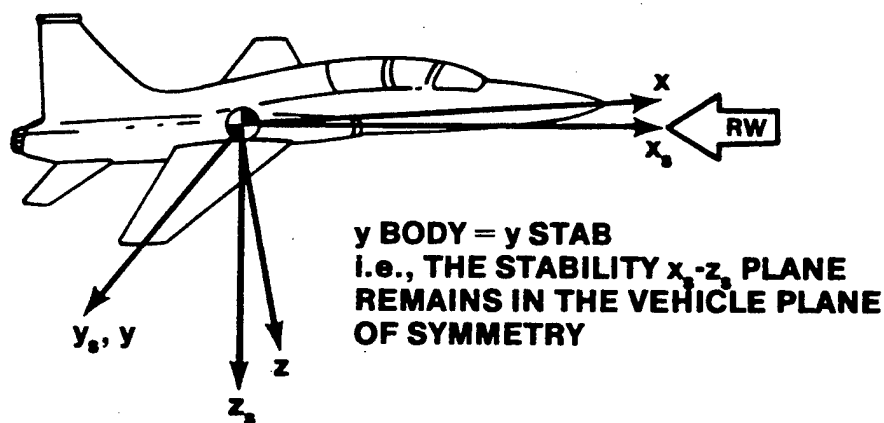


FIGURE 4.5. STABILITY AXIS SYSTEM

symmetric, i.e. with sideslip, then the  $x_s$ -axis is chosen to lie on the projection of the true velocity ( $\bar{V}_T$ ) in the plane of symmetry, with  $z_s$  also in the plane of symmetry. The moment of inertia and product of inertia terms vary for each equilibrium flight condition. They are assumed constant, however, in the equations of motion.

4.4.3.3 Principal Axes. These are a special set of body axes aligned with the principal axes of the vehicle and are used for certain applications. Principal axes are defined as those axes where all of the products of inertia are reduced to zero. The equations of motion are thus greatly simplified, but it is difficult to accurately describe the aircraft motion in this system.

4.4.3.4 Wind Axes. The wind axes use the vehicle translational velocity as the reference for the axis system. Wind axes are thus oriented with respect to the flight path of the vehicle, i.e., with respect to the relative wind,  $\bar{V}_T$ . If the reference flight condition is symmetric, i.e.,  $\bar{V}_T$  lies in the vehicle plane of symmetry, then the wind axes coincide with the stability axes. The wind axes depart from the stability axes, moving with the relative wind, when sideslip is present.

The relationship between the wind axes and the vehicle body axes of a rigid body defines the angle of attack,  $\alpha$ , and the sideslip angle,  $\beta$ . These angles are convenient independent variables for use in the expression of aerodynamic force and moment coefficients.

Wind axes are not generally used in the analysis of the motion of a rigid body, because, as in the case of the earth axes, the moment of inertia and product of inertia terms in the three rotational equations of motion vary with time,  $\alpha$ , and  $\beta$ .

#### 4.5 DERIVATION OF THE RIGHT HAND SIDE (RHS) OF THE EQUATIONS OF MOTION

The RHS of the equation represents the aircraft response to forces or moments. Through the application of Newton's second law, two vector relations can be used to derive the six required equations, three translational and three rotational. The actual aircraft will be flexible, which gives rise to aeroelastic effects and are additional degrees of freedom (requiring more variables and equations). These will be considered separately (in Chapter 12).

In order to derive the equations of motion we will make the following

ASSUMPTION: The aircraft is a rigid body.

As noted earlier, Newton's second law is valid only in an inertial coordinate system. For most aircraft, the fixed earth axis system can be assumed to be an inertial coordinate system. In order to do this, we make the following

ASSUMPTION: The earth and atmosphere are fixed in inertial space.

In addition, most motion of interest in stability and control takes place in a relatively short time. We can also, therefore, usually make the following

ASSUMPTION: Mass ( $m$ ) is constant ( $dm/dt = 0$ ).

#### 4.5.1 Translational Force Relations

The vector equation for the aircraft translation from Newton's second law (eq. 4.1) is

$$\bar{F} = \frac{d(m\bar{V}_T)}{dt} \Big|_{xyz} \quad (4.9)$$

where  $\bar{V}_T$  is the true velocity of the aircraft. Figure 4.6 shows how this vector changes in both magnitude and direction with respect to the  $xyz$  (body) and  $XYZ$  (fixed earth) axes.

From vector analysis, the derivative of the velocity  $\bar{V}_T$  in the inertial (fixed earth) coordinate system is related to the derivative of  $\bar{V}_T$  in the body axis system through the relationship

$$\frac{d\bar{V}_T}{dt} \Big|_{XYZ} = \frac{d\bar{V}_T}{dt} \Big|_{xyz} + \bar{\omega} \times \bar{V}_T \quad (4.10)$$

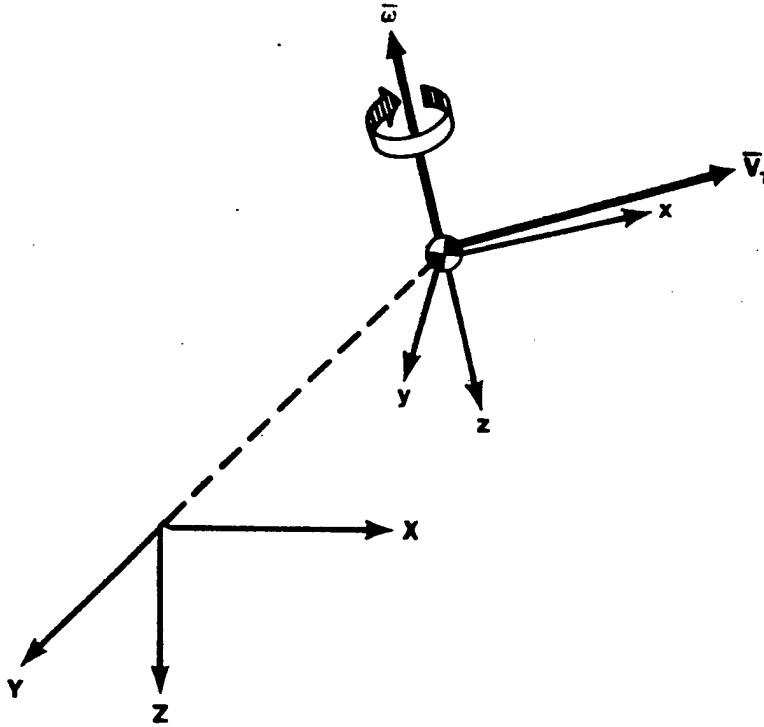


FIGURE 4.6. TRUE VELOCITY IN BODY AND FIXED EARTH AXES COORDINATE SYSTEMS

Substituting this into Equation 4.9 (and assuming mass is constant), the applied force is

$$\bar{F} = m \left[ \left. \frac{d\bar{V}_r}{dt} \right|_{xyz} + \bar{\omega} \times \bar{V}_r \right] \quad (4.11)$$

$\bar{V}_r$  and  $\bar{\omega}$  are two of the four vectors used in the equations of motion to describe the vehicle motion ( $\bar{F}$  and  $\bar{G}$  are the other two). They are defined as follows:

$$\bar{V}_r = U\bar{i} + V\bar{j} + W\bar{k} \quad (4.12)$$

where

U = forward velocity  
V = side velocity  
W = vertical velocity

and

$$\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k} \quad (4.13)$$

where

P = roll rate  
Q = pitch rate  
R = yaw rate

The relationship of the true velocity and its components to  $\alpha$  and  $\beta$  and the body axis coordinate system is shown in Figure 4.7.

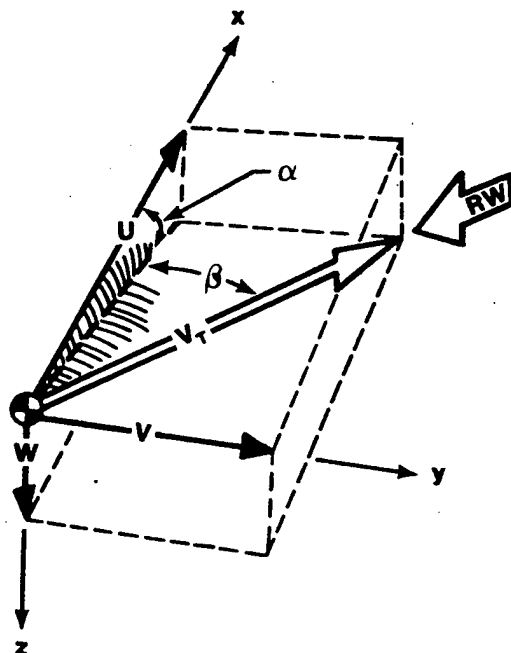


FIGURE 4.7. VELOCITY COMPONENTS AND THE AERODYNAMIC ORIENTATION ANGLES,  $\alpha$  AND  $\beta$

The angles  $\alpha$  and  $\beta$  can be expressed in terms of the velocity components as follows:

$$\sin \alpha = \frac{W}{V_T \cos \beta} \quad (4.14)$$

If  $\beta$  is small (ASSUMPTION), then  $\cos \beta \approx 1$  and

$$\sin \alpha \approx \frac{W}{V_T} \quad (4.15)$$

If  $\alpha$  is also small (ASSUMPTION), then

$$\alpha \approx \frac{W}{V_T} \quad (4.16)$$

For angle of sideslip

$$\sin \beta = \frac{V}{V_T} \quad (4.17)$$

If  $\beta$  is small, then  $\sin \beta \approx \beta$  and

$$\beta \approx \frac{V}{V_T} \quad (4.18)$$

Using equations 4.12 and 4.13, the translational equation (4.11) can now be written in component form as

$$\bar{F} = m [ \dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ U & V & W \end{vmatrix} ] \quad (4.19)$$

Expanding

$$\bar{F} = m [ \dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + (QW - RV)\bar{i} - (PW - RU)\bar{j} + (PV - QU)\bar{k} ] \quad (4.20)$$

Rearranging

$$\bar{F} = m [ (\dot{U} + QW - RV)\bar{i} + (\dot{V} + RU - PW)\bar{j} + (\dot{W} + PV - QU)\bar{k} ] \quad (4.21)$$

In component form, the sum of forces in the body axis system is

$$\bar{F} = F_x \bar{i} + F_y \bar{j} + F_z \bar{k} \quad (4.22)$$

this results in three component translational equations:

$$F_x = m (\dot{U} + QW - RV) \quad (4.23)$$

$$F_y = m (\dot{V} + RU - PW) \quad (4.24)$$

$$F_z = m (\dot{W} + PV - QU) \quad (4.25)$$

#### 4.5.2 Rotational Equations

Once again from Newton's second law (eq. 4.2),

$$\bar{G} = \left. \frac{d(\bar{H})}{dt} \right|_{XYZ} \quad (4.26)$$

Equation 4.26 states the change in angular momentum,  $\bar{H}$ , is equal to the applied moments,  $\bar{G}$ .

Angular momentum should not be as difficult to understand as some people would like to make it. It can be thought of as linear momentum with a moment arm included. Consider a ball swinging on the end of a string, at any instant of time, as shown in Figure 4.8.

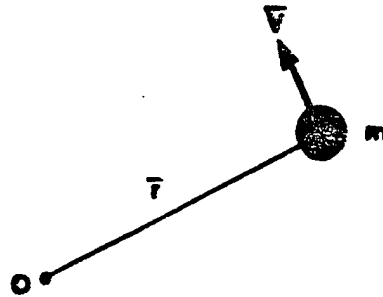


FIGURE 4.8. ANGULAR MOMENTUM

The linear momentum of this system would be:

$$\overline{\text{Linear Momentum}} = m\bar{v}$$

Angular momentum is defined as  $\bar{H}$ , where  $\bar{H} = \bar{r} \times \overline{\text{Linear Momentum}}$  and, since in the example of Figure 4.8, the angle between  $\bar{r}$  and  $\bar{v}$  is 90 degrees, the magnitude of the angular momentum is  $mrV$ .

Just as a force  $\bar{F}$  changes linear momentum, ( $\bar{F} = \frac{d}{dt} m\bar{v}$ ), a moment  $\bar{G}$  will change angular momentum ( $\bar{G} = \frac{d}{dt} \bar{H}$ ). A moment is related to a force in the same manner that angular momentum is related to linear momentum:

$$\overline{\text{Moment}} = \bar{r} \times \overline{\text{Force}}$$

$$\overline{\text{Angular Momentum}} = \bar{r} \times \overline{\text{Linear Momentum}}$$

In order for us to determine the angular momentum of the aircraft, consider a small element of mass  $m_1$ , somewhere in the aircraft, a distance  $\bar{r}_1$  from the cg (Figure 4.9).

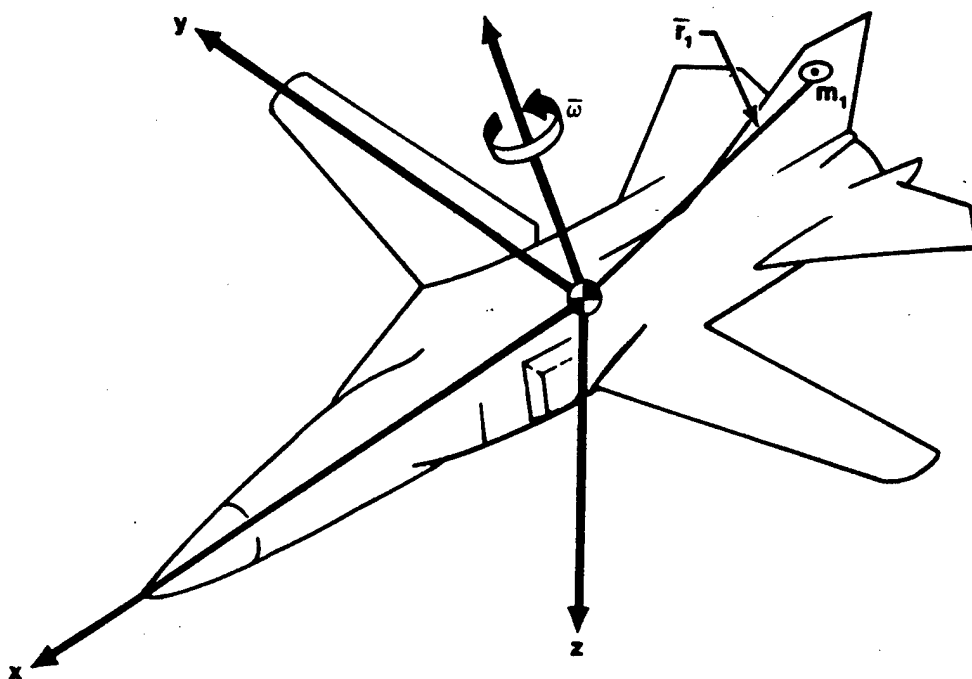


FIGURE 4.9. ELEMENTAL DEVELOPMENT OF RIGID BODY ANGULAR MOMENTUM

The angular momentum of  $m_1$  is

$$\bar{H}_{m_1} = \bar{r}_1 \times m\bar{v}_1 = m_1 (\bar{r}_1 \times \bar{v}_1) \quad (4.27)$$

and

$$\bar{v}_1 = \left. \frac{d\bar{r}_1}{dt} \right|_{xyz} \quad (\text{i.e., in the inertial coordinate system})$$

Again from vector analysis, the rate of change of the radius vector  $\bar{r}_1$  can be related to the body axis system (xyz) by

$$\bar{v}_1 = \left. \frac{d\bar{r}_1}{dt} \right|_{xyz} = \left. \frac{d\bar{r}_1}{dt} \right|_{xyz} + \bar{\omega} \times \bar{r}_1 \quad (4.29)$$

since the aircraft is a rigid body  $\bar{r}_1$  does not change with time (assuming no aeroelastic effects). Therefore, the first term can be excluded, and the inertial velocity of the element  $m_1$  is

$$\bar{V}_1 = \bar{\omega} \times \bar{r}_1 \quad (4.30)$$

Substituting this into Equation 4.27

$$\bar{H}_{m_1} = m_1 [\bar{r}_1 \times (\bar{\omega} \times \bar{r}_1)] \quad (4.31)$$

This is the angular momentum of the elemental mass  $m_1$ . In order to find the angular momentum of the whole aircraft, we integrate over the aircraft volume (V).  $\rho_A$  is the mass density of the aircraft.

$$\bar{H} = \int_V \rho_A [\bar{r} \times (\bar{\omega} \times \bar{r})] dV \quad (4.32)$$

where

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \quad (4.33)$$

then

$$\bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ x & y & z \end{vmatrix} \quad (4.34)$$

The determinant can be expanded to give

$$\bar{\omega} \times \bar{r} = (Qz - Ry)\bar{i} + (Rx - Pz)\bar{j} + (Py - Qx)\bar{k} \quad (4.35)$$

therefore, Equation 4.32 becomes

$$\bar{H} = \int_V \rho_A \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ (Qz-Ry) & (Rx-Pz) & (Py-Qx) \end{vmatrix} dV \quad (4.36)$$

So the components of  $\bar{H}$  are

$$H_x = \int_V \rho_A y(Py - Qx)dV - \int_V \rho_A z(Rx - Pz)dV \quad (4.37)$$

$$H_y = \int_V \rho_A z(Qz - Ry)dV - \int_V \rho_A x(Py - Qx)dV \quad (4.38)$$

$$H_z = \int_V \rho_A x(Rx - Pz)dV - \int_V \rho_A y(Qz - Ry)dV \quad (4.39)$$

Rearranging the equations

$$H_x = P \int_V \rho_A (y^2 + z^2)dV - Q \int_V \rho_A xydV - R \int_V \rho_A xzdV \quad (4.40)$$

$$H_y = Q \int_V \rho_A (x^2 + z^2) dV - R \int_V \rho_A yz dV - P \int_V \rho_A xy dV \quad (4.41)$$

$$H_z = R \int_V \rho_A (x^2 + y^2) dV - P \int_V \rho_A xz dV - Q \int_V \rho_A yz dV \quad (4.42)$$

The integrals are now recognizable as moments and products of inertia. The moments of inertia are defined as

$$I_x = \int_V \rho_A (y^2 + z^2) dV \quad (4.43)$$

$$I_y = \int_V \rho_A (x^2 + z^2) dV \quad (4.44)$$

$$I_z = \int_V \rho_A (x^2 + y^2) dV \quad (4.45)$$

These are a measure of resistance to rotation and are never zero. They are illustrated in Figure 4.10.

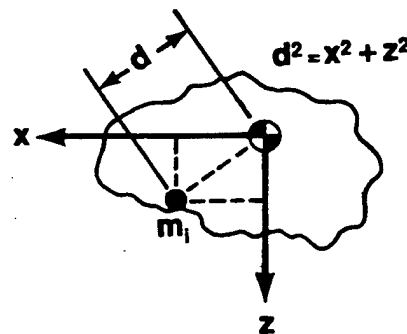


FIGURE 4.10. MOMENT OF INERTIA ( $I_y$ )

The products of inertia are defined as (Figure 4.11)

$$I_{xy} = I_{yx} = \int_V \rho_A xy dV \quad (4.46)$$

$$I_{yz} = I_{zy} = \int_V \rho_A yz dV \quad (4.47)$$

$$I_{xz} = I_{zx} = \int_V \rho_A xz dV \quad (4.48)$$

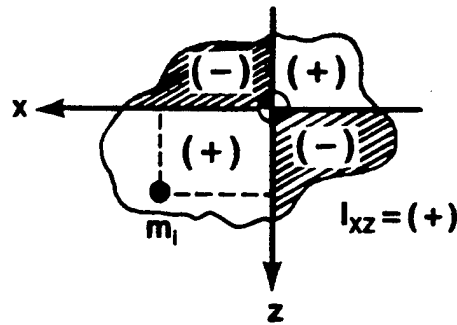


FIGURE 4.11. PRODUCT OF INERTIA ( $I_{xz}$ )

Products of inertia are measures of symmetry. They are zero for views having a plane of symmetry.

Substituting into equations 4.40 to 4.42, we find the angular momentum of a rigid body is

$$\bar{H} = H_x \bar{i} + H_y \bar{j} + H_z \bar{k} \quad (4.49)$$

So that

$$H_x = PI_x - QI_{xy} - RI_{xz} \quad (4.50)$$

$$H_y = QI_y - RI_{yz} - PI_{xy} \quad (4.51)$$

$$H_z = RI_z - PI_{xz} - QI_{yz} \quad (4.52)$$

An aircraft is normally symmetric about the xz-axis as illustrated in Figure 4.12. In order to simplify the RHS of the equations of motion, therefore, we normally make the following

ASSUMPTION: The xz-plane is a plane of symmetry.

This causes two products of inertia,  $I_{xy}$  and  $I_{yz}$  to be zero. These may be cancelled out of the equations of motion. This restriction of the equations of motion (xz-plane symmetry) can be easily removed by including these terms. With the assumption, the angular momentum of a symmetric aircraft simplifies to

$$\bar{H} = (PI_x - RI_{xz}) \bar{i} + QI_y \bar{j} + (RI_z - PI_{xz}) \bar{k} \quad (4.53)$$

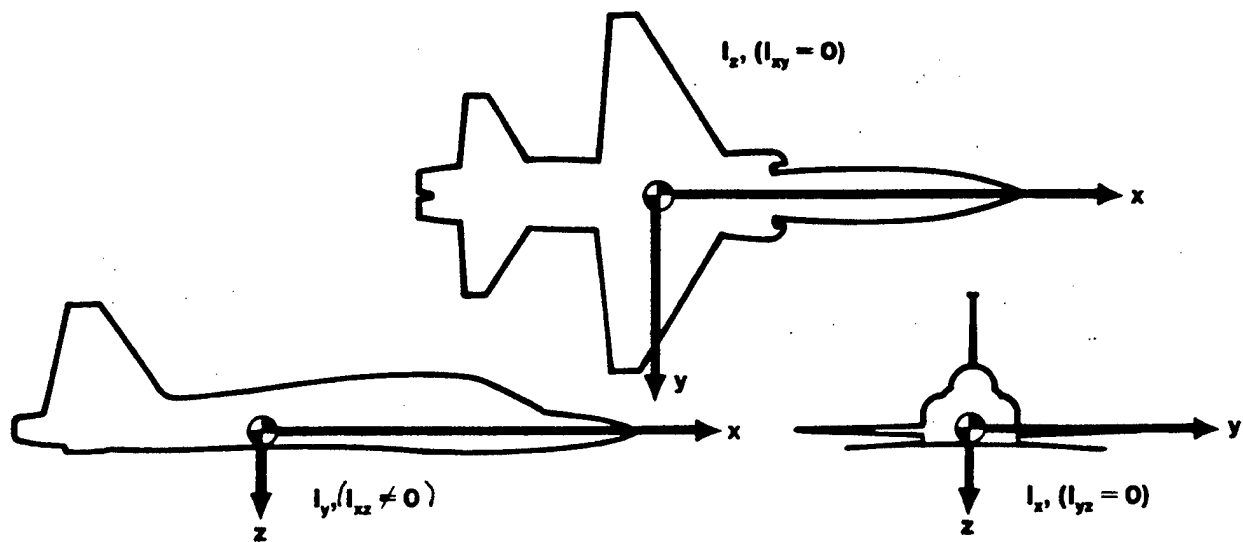


FIGURE 4.12. AIRCRAFT INERTIAL PROPERTIES WITH AN x-z PLANE OF SYMMETRY

The equation for angular momentum can now be substituted into the moment equation. Remember

$$\bar{G} = \left. \frac{d\bar{H}}{dt} \right|_{XYZ} \quad (4.54)$$

applies only with respect to inertial space. Expressed in the fixed body axis system, the equation becomes:

$$\bar{G} = \left. \frac{d\bar{H}}{dt} \right|_{xyz} + \bar{\omega} \times \bar{H} \quad (4.55)$$

which is

$$\bar{G} = \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ H_x & H_y & H_z \end{vmatrix} \quad (4.56)$$

Remember, for a symmetric aircraft,

$$\bar{H} = (PI_x - RI_{xz}) \bar{i} + QI_y \bar{j} + (RI_z - PI_{xz}) \bar{k} \quad (4.57)$$

Since the body axis system is used, the moments of inertia and the products of inertia are constant. Therefore, by differentiating and substituting, the moment equation becomes

$$\bar{G} = (\dot{P}I_x - \dot{R}I_{xz}) \bar{i} + \dot{Q}I_y \bar{j} + (\dot{R}I_z - \dot{P}I_{xz}) \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ (PI_x - RI_{xz}) & QI_y & (RI_z - PI_{xz}) \end{vmatrix} \quad (4.58)$$

Therefore, the rotational component equations are,

$$G_x = \dot{P}I_x + QR(I_z - I_y) - (\dot{R} + PQ)I_{xz} \quad (4.59)$$

$$G_y = \dot{Q}I_y - PR(I_z - I_x) + (P^2 - R^2)I_{xz} \quad (4.60)$$

$$G_z = \dot{R}I_z + PQ(I_y - I_x) + (QR - \dot{P})I_{xz} \quad (4.61)$$

This completes the development of the RHS of the six equations (equations 4.23 to 4.25, and 4.59 to 4.61).

#### 4.6 DERIVATION OF THE LHS OF THE EQUATIONS OF MOTION

The equations of motion relate the vehicle motion to the applied forces and moments:

LHS

RHS

Applied Forces and Moments = Observed Vehicle Motion

$$F_x = m(\dot{U} + QW - PV)$$

$$G_x = \dot{P}I_x + QR(I_z - I_y) - (\dot{R} + PQ)I_{xz}$$

etc.

The RHS of each of these six equations has been completely expanded in terms of easily measured quantities. The LHS must also be expanded in terms of convenient variables. In order to do this, we must be able to relate the orientation of the body axes (xyz) to the moving earth axes (XYZ). This is

done through the use of Euler angles. The moving earth axis system is used because we will be concerned with the orientation of the aircraft with respect to the earth and not its position (location of the cg) with respect to the earth.

#### 4.6.1 Euler Angles

The orientation of any coordinate system relative to another can be given by three angles (Euler angles), which are consecutive rotations about the z, y, and x axes, in that order, that carry one frame into coincidence with the other. In flight dynamics, the Euler angles used are those which rotate the vehicle carried moving earth axis system into coincidence with the relevant vehicle axis system (Figure 4.13).

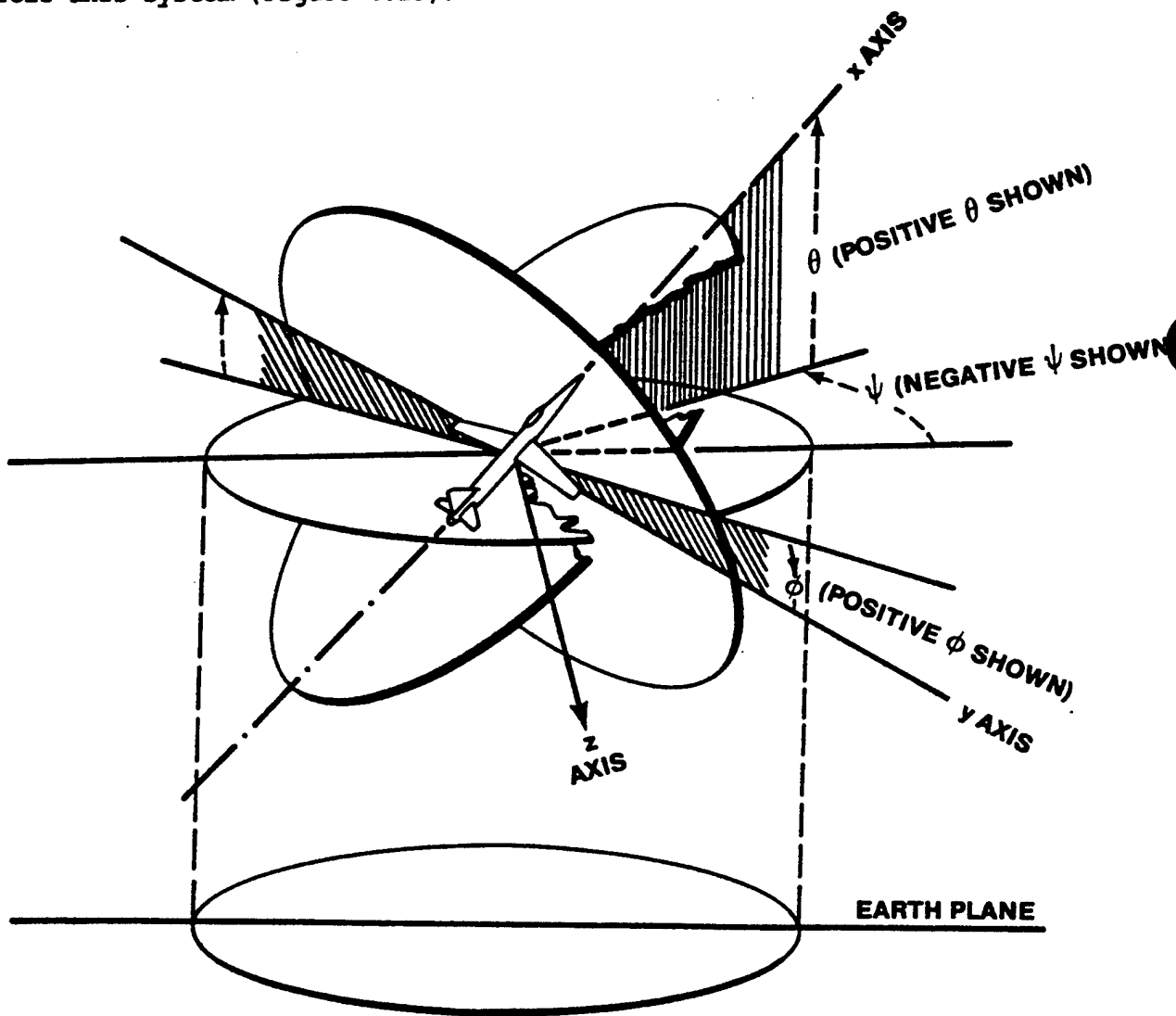


FIGURE 4.13. THE EULER ANGLE ROTATIONS

Euler angles are expressed as YAW ( $\psi$ ), PITCH ( $\theta$ ), and ROLL ( $\phi$ ). The sequence (YAW, PITCH, ROLL) must be maintained to arrive at the proper set of Euler angles. The Euler angles are defined as follows:

$\psi$  - Yaw Angle - The angle between the projection of x vehicle axes onto the horizontal plane and the initial reference position of the X earth axis. (Yaw angle is the vehicle heading only if the initial reference is North).

$\theta$  - Pitch Angle - The angle measured in a vertical plane between the x vehicle axis and the horizontal plane.

$\phi$  - Roll Angle - The angle, measured in the yz plane of the vehicle system, between the y axis and the horizontal plane. This is the same as bank angle for a given  $\psi$  and  $\theta$ , and is a measure of the rotation about the x axis to put the aircraft in the desired position from a wing's horizontal condition.

The accepted limits on the Euler angles are:

$$-180^\circ \leq \psi \leq +180^\circ$$

$$-90^\circ \leq \theta \leq +90^\circ$$

$$-180^\circ \leq \phi \leq +180^\circ$$

The importance of the sequence of the Euler angle rotations cannot be overemphasized. Finite angular displacements do not behave as vectors. Therefore, if the sequence is performed in a different order than  $\psi$ ,  $\theta$ ,  $\phi$ , the final result will be different. This fact is clearly illustrated by the final aircraft attitudes in Figure 4.14 in which two rotations of equal magnitude have been performed about the x and y axes, but in opposite order. Addition of a rotation about a third axis does nothing to improve the outcome.

Euler angles are very useful in describing the orientation of flight vehicles with respect to inertial space. Consequently, angular rates in an inertial system ( $\dot{\phi}$ ,  $\dot{\theta}$ ,  $\dot{\psi}$ ) can be transformed to angular rates in the vehicle axes (P, Q, R) using Euler angle transformations as developed in the next subsection.

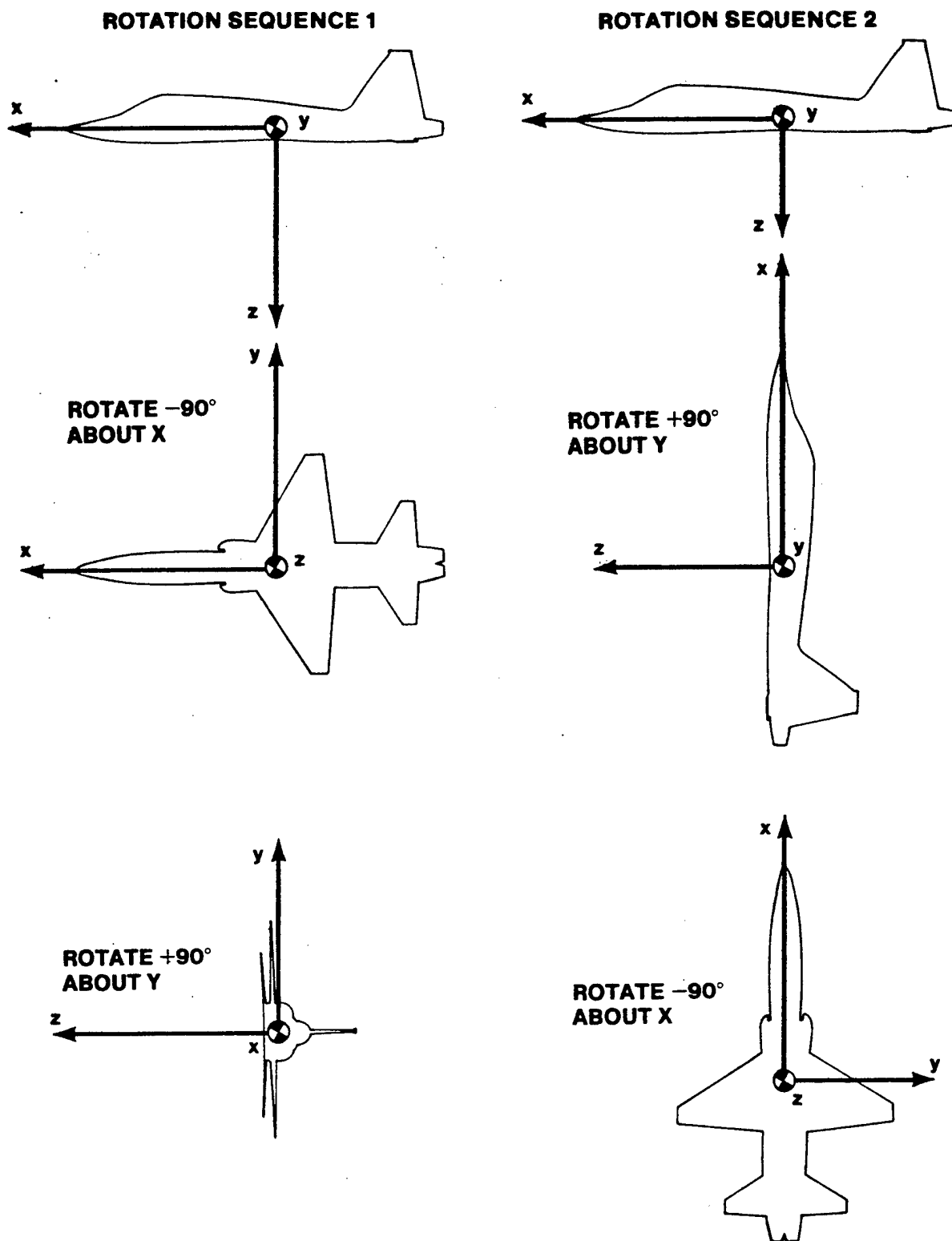


FIGURE 4.14. DEMONSTRATION THAT FINITE ANGULAR DISPLACEMENTS DO NOT BEHAVE AS VECTORS

#### 4.6.2 Angular Velocity Transformations

We need to develop equations to transform the angular rates from the moving earth axis system ( $\dot{\psi}$ ,  $\dot{\theta}$ ,  $\dot{\phi}$ ) into angular rates about the vehicle axis system (P, Q, R) for any aircraft attitude. The derivation (by vector resolution) is presented in the following paragraphs.

It is easy to see that when an aircraft is pitched up and banked, the vector  $\dot{\psi}$  will have components along the x, y, and z body axes (Figure 4.15). Remember,  $\dot{\psi}$  is the angular velocity about the Z axis of the moving earth axis system (it can be thought of as the rate of change of aircraft heading). Although it is not shown in Figure 4.15, the aircraft may have a value of  $\dot{\theta}$  and  $\dot{\phi}$ . In order to derive the transformation equations, it is easier to analyze one vector at a time. First resolve the components of  $\dot{\psi}$  on the body axes. Then do the same with  $\dot{\theta}$  and  $\dot{\phi}$ . The components can then be added and the total transformation will result.

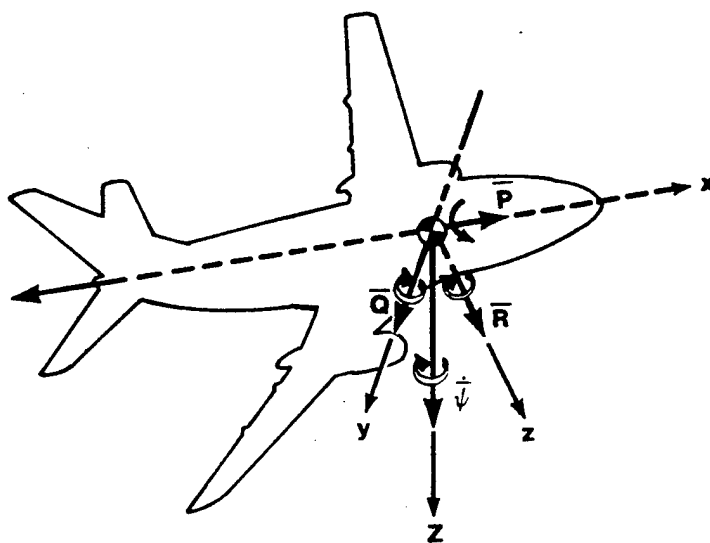


FIGURE 4.15. COMPONENTS OF  $\dot{\psi}$  ALONG x, y, AND z BODY AXES.  
(NOTE: THE X AND Y AXES OF THE MOVING EARTH  
AXIS SYSTEM ARE NOT SHOWN.)

Step 1 - Resolve the components of  $\dot{\psi}$  along the body axes for any aircraft attitude.

It is easy to see how  $\dot{\psi}$  reflects to the body axis by starting with an aircraft in straight and level flight and changing the aircraft attitude one angle at a time. In keeping with convention, the sequence of change will

be yaw, pitch and bank.

First, it can be seen from Figure 4.16 that the Z-axis of the moving earth axis system remains aligned with the z-axis of the body axis system regardless of the angle  $\psi$  if  $\theta$  and  $\phi$  are zero; therefore,  $\dot{\psi}$  does not affect P, and Q.

$$\therefore R = \dot{\psi} \text{ (when } \theta = \phi = 0 \text{)}$$

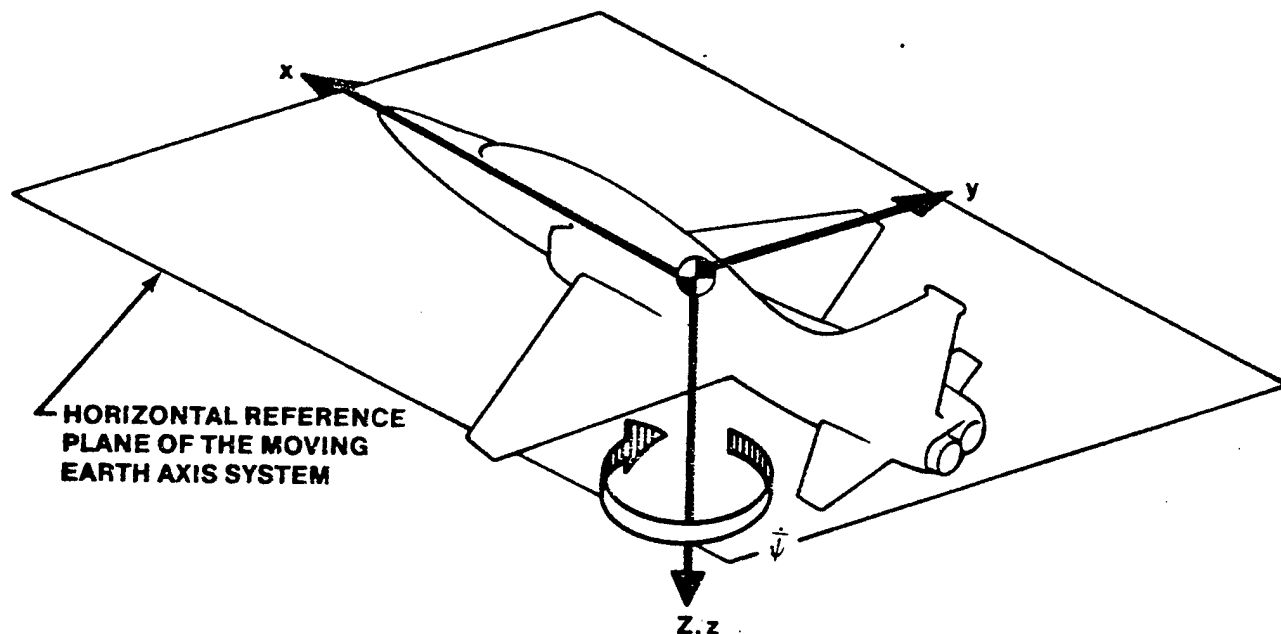


FIGURE 4.16. DEVELOPMENT OF AIRCRAFT ANGULAR VELOCITIES BY THE EULER ANGLE YAW RATE ( $\psi$  ROTATION)

Next, consider pitch up. In this attitude,  $\dot{\psi}$  has components on the x and z-body axes as shown in Figure 4.17. As a result,  $\dot{\psi}$  will contribute to the angular rates about these axis.

$$P = -\dot{\psi} \sin \theta \quad (4.62)$$

$$R = \dot{\psi} \cos \theta \quad (4.63)$$

With just pitch, the Z-axis remains perpendicular to the y-body axis, so Q is not affected by  $\dot{\psi}$  in this attitude.

Next, bank the aircraft, leaving the pitch as it is (Figure 4.18).

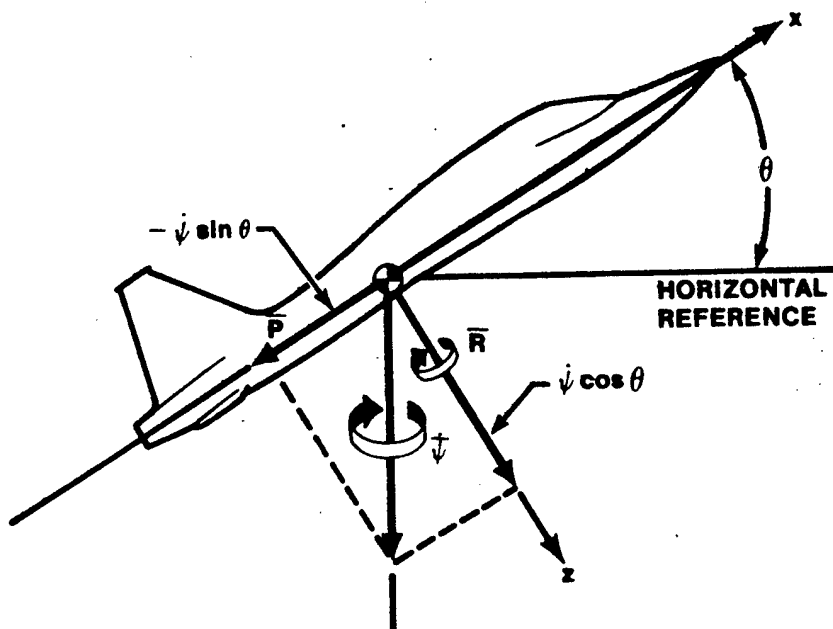


FIGURE 4.17. DEVELOPMENT OF AIRCRAFT ANGULAR VELOCITIES BY THE EULER ANGLE YAW RATE ( $\theta$  ROTATION)

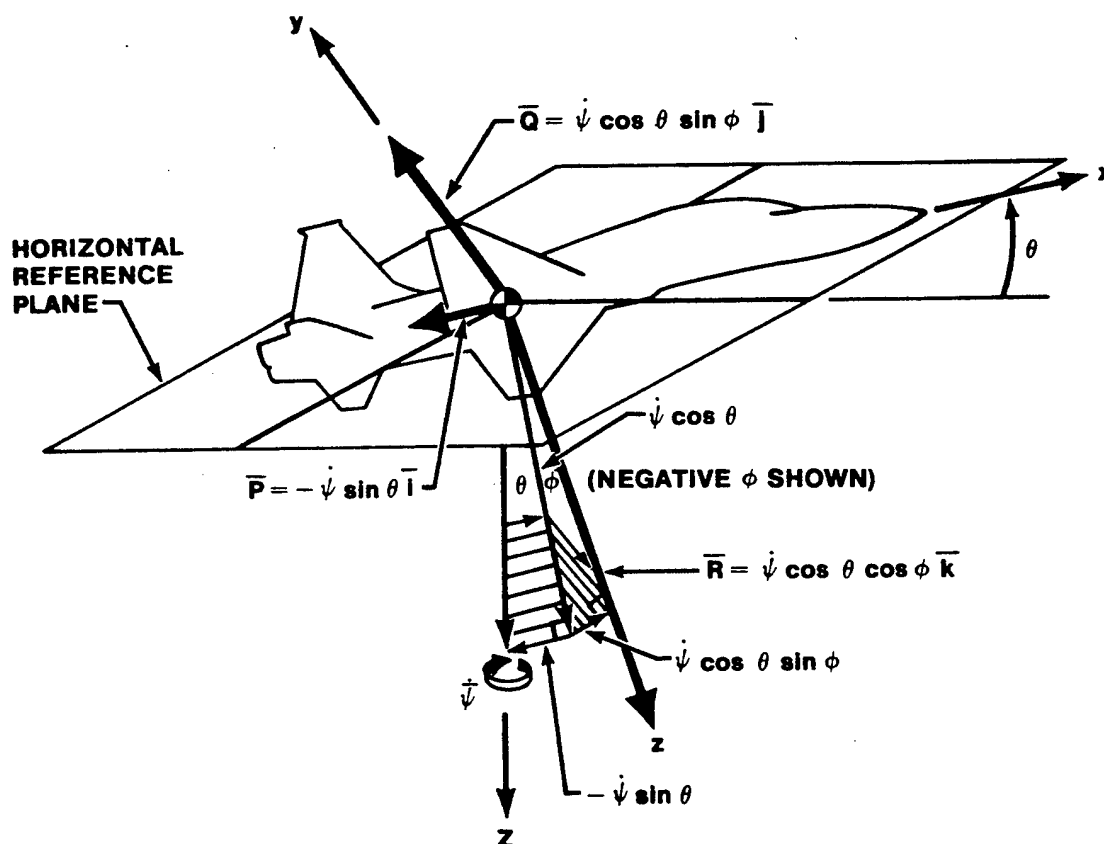


FIGURE 4.18. DEVELOPMENT OF AIRCRAFT ANGULAR VELOCITIES BY THE EULER ANGLE YAW RATE ( $\phi$  ROTATION)

All of the components are now illustrated. Notice that roll did not change the effect of  $\dot{\psi}$  on P. The components, therefore, of  $\dot{\psi}$  in the body axes for any aircraft attitude are

$$\begin{aligned} P &= -\dot{\psi} \sin \theta & (4.64) \\ Q &= \dot{\psi} \cos \theta \sin \phi & (4.65) \\ R &= \dot{\psi} \cos \theta \cos \phi & (4.66) \end{aligned}$$

(Effect of  $\dot{\psi}$  only)

Step 2 - Resolve the components of  $\dot{\theta}$  along the body axes for any aircraft attitude.

Remember,  $\theta$  is the angle between the x-body axis and the local horizontal (Figure 4.19). Once again, change the aircraft attitude by steps in the sequence of yaw, pitch, and bank and analyze the effects of  $\dot{\theta}$ .

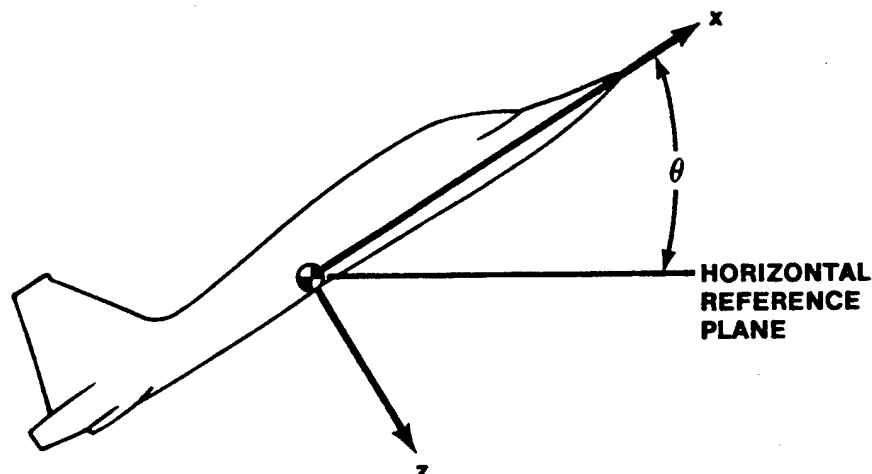


FIGURE 4.19. CONTRIBUTION OF THE EULER PITCH ANGLE RATE TO AIRCRAFT ANGULAR VELOCITIES ( $\theta$  ROTATION)

It can be seen immediately that the yaw angle has no effect. Likewise when pitched up, the y-body axis remains in the horizontal plane. Therefore,  $\dot{\theta}$  is the same as  $\bar{Q}$  in this attitude and the component is equal to

$$Q = \dot{\theta}$$

Now bank the aircraft.

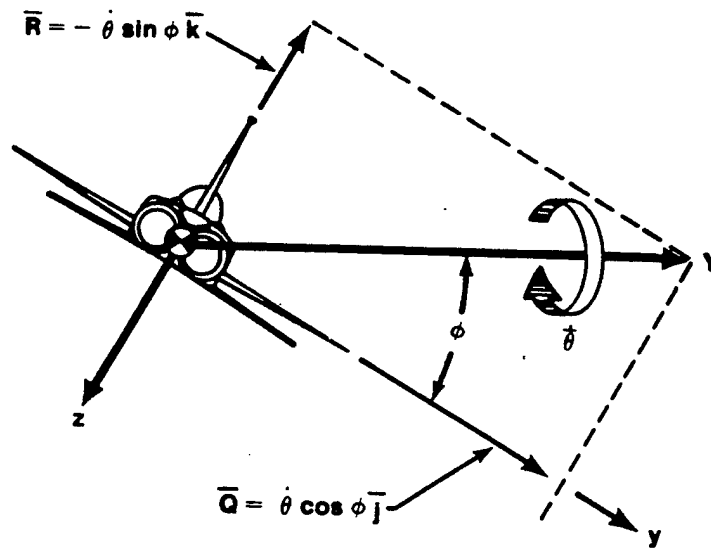


FIGURE 4.20. CONTRIBUTION OF THE EULER PITCH ANGLE RATE TO AIRCRAFT ANGULAR VELOCITIES ( $\phi$  ROTATION)

It can be seen from Figure 4.20 that the components of  $\dot{\theta}$  on the body axes are

$$Q = \dot{\theta} \cos \phi \quad (4.67)$$

$$R = -\dot{\theta} \sin \phi \quad (4.68)$$

Notice that  $P$  is not affected by  $\dot{\theta}$  since by definition  $\dot{\theta}$  is measured on an axis perpendicular to the  $x$  body axis.

Step 3 - Resolve the components of  $\dot{\phi}$  along the body axes.

This one is easy since by definition  $\dot{\phi}$  is measured along the  $x$  body axis. Therefore,  $\dot{\phi}$  affects the value of  $P$  only, or

$$P = \dot{\phi} \quad (4.69)$$

The components of  $\dot{\psi}$ ,  $\dot{\theta}$ , and  $\dot{\phi}$  along the  $x$ ,  $y$ , and  $z$  body axes for any aircraft attitude have been derived. These can now be summed to give the transformation equations.

$$\bar{P} = (\dot{\phi} - \dot{\psi} \sin \theta) \bar{I} \quad (4.70)$$

$$\bar{Q} = (\dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi) \bar{J} \quad (4.71)$$

$$\bar{R} = (\dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi) \bar{K} \quad (4.72)$$

With these equations it is now possible to transform the equations of motion written in body axis terms ( $U, V, W, P, Q,$  and  $R$ ) in terms of the motion seen in the inertial (earth axis) system ( $U, V, W, \psi, \theta,$  and  $\phi$ ). In that case, the resulting equations are six simultaneous nonlinear differential equations that are of the first order in  $U, V,$  and  $W$  and of the second order in  $\psi, \theta,$  and  $\phi$ . In order to completely describe the trajectory of the aircraft in the inertial coordinate system, a similar transformation is required to relate  $U, V,$  and  $W$  to the velocities in the inertial coordinate system (not covered in this text).

Equations 4.70 to 4.72 are known as the parametric equations and they, along with the six equations of motion, can be used to describe the complete motion of the aircraft.

#### 4.6.3 Initial Breakdown of the LHS

In general, the applied forces and moments on the LHS can be broken up according to the sources shown below.

		SOURCE					
		Aero-dynamic	Direct Thrust	Gravity	Gyro-Scopic	Other	
LONGITUDINAL	$F_x$	$X_A$	$X_T$	$X_g$	0	$X_{oth}$	$= m\dot{U} + \dots \quad (4.73)$
	$F_z$	$Z_A$	$Z_T$	$Z_g$	0	$Z_{oth}$	$= m\dot{W} + \dots \quad (4.73a)$
	$G_y$	$M_A$	$M_T$	0	$M_{gyro}$	$M_{oth}$	$= \dot{Q}I_y + \dots \quad (4.74)$
LATERAL-DIRECTIONAL	$F_y$	$Y_A$	$Y_T$	$Y_g$	0	$Y_{oth}$	$= m\dot{V} + \dots \quad (4.74a)$
	$G_x$	$L_A$	$L_T$	0	$L_{gyro}$	$L_{oth}$	$= \dot{P}I_x + \dots \quad (4.75)$
	$G_z$	$N_A$	$N_T$	0	$N_{gyro}$	$N_{oth}$	$= \dot{R}I_z + \dots \quad (4.75a)$

1. Aerodynamic Forces and Moments - These will be further expanded into stability parameters and derivatives (discussed in subsection 4.6.4).
2. Direct Thrust Forces and Moments - These terms include the effect of the thrust vector itself - they usually do not include the indirect or induced effects of jet flow or running propellers (discussed in subsection 4.6.5).
3. Gravity Forces - These vary with orientation of the gravity vector (discussed in subsection 4.6.6).
4. Gyroscopic Moments - These occur as a result of large rotating masses such as engines and props (discussed in section 4.6.7).
5. Other Sources - These include spin chutes, reaction controls, etc. (not discussed in this chapter).

#### 4.6.4 Aerodynamic Forces And Moments

By far the most important forces and moments on the LHS of the equation are the aerodynamic terms. Unfortunately, they are also the most complex. As a result, certain simplifying assumptions are made, and several of the smaller terms are arbitrarily excluded to simplify the analysis. Remember we are not trying to design an aircraft around some critical criteria. We are only trying to derive a set of equations that will help us analyze the important factors affecting aircraft stability and control.

4.6.4.1 Choice Of Axis System. Consider only the aerodynamic forces on an aircraft. Summing forces along the x body axis (Figure 4.21)

$$F_x = L \sin \alpha - D \cos \alpha \quad (4.76)$$

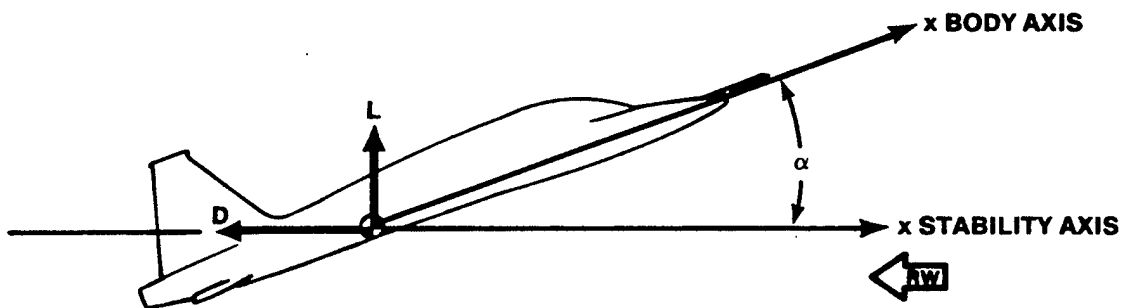


FIGURE 4.21. CHOICE OF AXIS SYSTEM

Notice that if the forces were summed along the  $x_s$  stability axis (Figure 4.21), it would be

$$F_x = -D \quad (4.77)$$

It would simplify things if the stability axes were used for development of the aerodynamic forces. A small angle assumption enables us to do this:

$$\cos \alpha \approx 1$$

$$\sin \alpha \approx 0$$

Using this assumption, equation 4.76 reduces to equation 4.77. Whether it be thought of as a small angle assumption or as an arbitrary choice of the stability axis system, the result is less complexity. This would not be done for preliminary design analyses; however, for the purpose of deriving a set of equations to be used as an analytical tool in determining handling qualities, the assumption is perfectly valid, and is surprisingly accurate for relatively large values of  $\alpha$ . It should be noted that lift and drag are defined to be positive as illustrated. Thus these quantities have a negative sense with respect to the stability axis system.

The aerodynamic terms will be developed using the stability axis system so that the equations assume the form,

$$\text{"DRAG"} \quad -D + X_T + X_g + X_{oth} = m\dot{U} + - - - - \quad (4.78)$$

$$\text{"LIFT"} \quad -L + Z_T + Z_g + Z_{oth} = m\dot{W} + - - - - \quad (4.79)$$

$$\text{"PITCH"} \quad M_A + M_T + M_{gyro} + M_{oth} = \dot{Q}I_y + - - - - \quad (4.80)$$

$$\text{"SIDE"} \quad Y_A + Y_T + Y_g + Y_{oth} = m\dot{V} + - - - - \quad (4.81)$$

$$\text{"ROLL"} \quad L_A + L_T + L_{gyro} + L_{oth} = \dot{P}I_x + - - - - \quad (4.82)$$

$$\text{"YAW"} \quad N_A + N_T + N_{gyro} + N_{oth} = \dot{R}I_z + - - - - \quad (4.83)$$

4.6.4.1A Coordinate Systems and Transformations. The five orthogonal coordinate systems are related by the figure below. In general, you need only to rotate through the given angles to transfer from one set of axes to another.

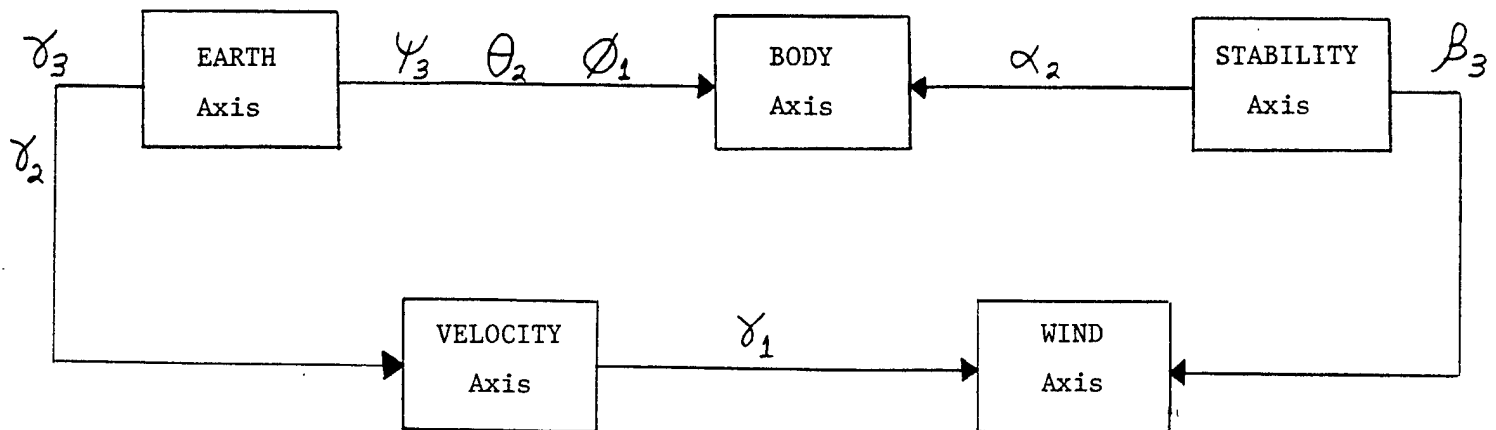


FIGURE 4.21A AXIS SYSTEM RELATIONSHIPS

It is often convenient to measure forces or moments in a certain axis system while the equations of motion are better understood in another. If you are transforming in the direction indicated, use the normal rotation matrices. If transforming opposite to the indicated direction use the transpose of the rotation matrices. In either case put in the measured angle(s) without changing sign(s). For example, weight is measured easily in the earth axis system. To transform to the body axis the process is to pre-multiply the weight vector(earth axis) by  $R_3$  then  $R_2$  and then  $R_1$  as shown.

$$\begin{bmatrix} \text{Weight} \end{bmatrix}_{\text{Body}} = R_1(\phi_1) R_2(\theta_2) R_3(\psi_3) \begin{bmatrix} 0 \\ 0 \\ \text{Weight} \end{bmatrix}_{\text{Earth}}$$

Where the rotation matrices are

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(*) & \sin(*) \\ 0 & -\sin(*) & \cos(*) \end{bmatrix} \quad R_2 = \begin{bmatrix} \cos(*) & 0 & -\sin(*) \\ 0 & 1 & 0 \\ \sin(*) & 0 & \cos(*) \end{bmatrix} \quad R_3 = \begin{bmatrix} \cos(*) & \sin(*) & 0 \\ -\sin(*) & \cos(*) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the (\*) represents the appropriate angle from figure 4.21A

4.6.4.2 Expansion of Aerodynamic Terms. A stability and control analysis is concerned with how a vehicle responds to perturbation inputs. For instance, up elevator should cause the nose to come up; or for turbulence caused sideslip, the aircraft should realign itself with the relative wind. Intuitively, the aerodynamic terms have the most effect on the resulting motion of the aircraft. Unfortunately, the above equations (that result from summing forces and moments), are non-linear, and exact solutions are impossible. In view of the complexity of the problem, linearization of the equations brings about especially desirable simplifications. The linearized model is based on the assumption of small disturbances and the small perturbation theory. This model, nonetheless, gives quite adequate results for engineering purposes over a wide range of applications; because the major aerodynamic effects are nearly linear functions of the variables of interest, and because quite large disturbances in flight may correspond to relatively small disturbances in the linear and angular velocities.

4.6.4.3 Small Perturbation Theory. The small perturbation theory is based on a simple technique used for linearizing a set of differential equations. In aircraft flight dynamics, the aerodynamic forces and moments are assumed to be functions of the instantaneous values of the perturbation velocities, control deflections, and of their derivatives. They are obtained in the form of a Taylor series in these variables, and the expressions are linearized by excluding all higher-order terms. To fully understand the derivation, some assumptions and definitions must first be established.

4.6.4.3.1 The Small Disturbance Assumption - A summary of the major variables that affect the aerodynamic characteristics of a rigid body or a vehicle is given below.

1. Velocity, temperature, and altitude: These variables may be considered directly or indirectly functions of Mach, Reynolds number, and dynamic pressure. Velocity may be resolved into components  $U$ ,  $V$ , and  $W$  along the vehicle body axes.
2. Angle of attack,  $\alpha$ , and angle of sideslip,  $\beta$ : These variables may be used with the magnitude of the total velocity,  $V_T$ , to express the orthogonal velocity components  $U$ ,  $V$ , and  $W$ . It is more convenient to express variation of force and moment characteristics with these angles as independent variables rather than the velocity components.

3. Angular velocity: This is usually resolved into components, P, Q, and R about the vehicle body axes.
4. Control surface deflections: These are used primarily to change or balance aerodynamic forces and moments, and are accounted for by  $\delta_e$ ,  $\delta_a$ ,  $\delta_r$  (the elevator, aileron and rudder deflection, respectively).

Because air has mass, the flow field cannot adjust instantaneously to sudden changes in these variables, and transient conditions exist. In some cases, these transient effects become significant. Analysis of certain unsteady motions may therefore require consideration of the time derivatives of the variables listed above. In other words:

		VARIABLE	FIRST DERIVATIVE	SECOND DERIVATIVE
$\left. \begin{array}{c} D \\ L \\ M_A \\ Y_A \\ L_A \\ N_A \end{array} \right\}$	$\left. \begin{array}{c} \text{Are a} \\ \text{Function} \\ \text{of} \end{array} \right\}$	U $\alpha$ $\beta$	$\dot{U}$ $\dot{\alpha}$ $\dot{\beta}$	$\ddot{U}$ $\ddot{\alpha}$ $\ddot{\beta}$
		P   Q   R	$\dot{P}$ $\dot{Q}$ $\dot{R}$	$\ddot{P}$ $\ddot{Q}$ $\ddot{R}$
		$\delta_e$ $\delta_a$ $\delta_r$	$\dot{\delta}_e$ $\dot{\delta}_a$ $\dot{\delta}_r$	$\ddot{\delta}_e$ $\ddot{\delta}_a$ $\ddot{\delta}_r$
		$\rho$ M $R_e$ T	assumed constant	- - - - -

This rather formidable list can be reduced to workable proportions by assuming that the vehicle motion consists only of small deviations from some initial reference condition. In addition, a Taylor series expansion, with higher order terms assumed negligible, is used to determine the effect of these small perturbations on the aircraft. Fortunately, this small disturbance assumption applies to many cases of practical interest and, as a bonus, stability parameters and derivatives derived under this assumption continue to give good results for somewhat larger motions.

The variables are considered to consist of some equilibrium value plus an incremental change, called the "perturbed value." The notation for these perturbed values is usually lower case. For example,

$$P = P_0 + p$$

$$U = U_0 + u$$

In summary, the small disturbance assumption is applied in three steps: assuming an initial (equilibrium) condition (described in subsection 4.6.4.3.3), assuming vehicle motion consists of small perturbations about this condition and, using a first order Taylor series expansion (described in subsection 4.6.4.3.4) to determine the effect of these small perturbations. As an additional consequence, the small perturbation assumption allows us to decouple the longitudinal and lateral-directional equations as discussed in the next subsection.

4.6.4.3.2 Longitudinal and Lateral-Directional Equations - It has been found from experience that, when operating under the small perturbation assumption, the vehicle motion can be thought of as two independent (decoupled) motions, each of which is a function only of the variables shown below.

1. Longitudinal Motion

$$(D, L, M_A) = f(U, \alpha, \dot{\alpha}, Q, \delta_e) \quad (4.84)$$

2. Lateral-Directional Motion

$$(Y_A, L_A, N_A) = f(\beta, \dot{\beta}, P, R, \delta_a, \delta_r) \quad (4.85)$$

The equations are grouped and named in the above manner because the state variables of the first group  $U, \alpha, \dot{\alpha}, Q, \delta_e$  are known as the longitudinal variables and those of the second group,  $\beta, \dot{\beta}, P, R, \delta_a$ , and  $\delta_r$ , are known as the lateral-directional variables. With the conventional simplifying assumptions, the longitudinal and lateral-directional variables will appear explicitly only in their respective group. This separation will also be displayed in the aerodynamic force and moment terms and the equations will completely decouple into two independent sets.

4.6.4.3.3 Initial Conditions - As stated earlier, we will assume that the motion consists of small perturbations about some initial equilibrium condition. The condition we will assume is steady straight symmetrical flight. This condition is a combination of the following motions:

Steady Flight. Motion with zero rates of change of the linear and angular velocity components, i.e.,

$$\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = 0.$$

Straight Flight. Motion with zero angular velocity components,  $P$ ,  $Q$ , and  $R = 0$ .

Symmetric Flight. Motion in which the vehicle plane of symmetry remains fixed in space throughout the maneuver. The unsymmetric variables  $P$ ,  $R$ ,  $V$ ,  $\phi$ , and  $\beta$  are all zero in symmetric flight. Some symmetric flight conditions are wings-level dives, climbs, and pull-ups with no sideslip. In steady straight symmetric flight, the aircraft is assumed to be flying wings level with all components of velocity zero except  $U_0$  and  $W_0$ . Therefore, with reference to the body axis

$$V_T \approx U_0 \approx \text{constant}$$

$$W_0 \approx \text{small constant} \therefore \alpha_0 \approx \text{small constant}$$

$$V_0 \approx 0 \therefore \beta_0 \approx 0$$

$$P_0 \approx Q_0 \approx R_0 \approx 0$$

We have already found that the equations of motion simplify considerably when the stability axis is used as the reference axis. This idea will again be employed and the final set of boundary conditions will result. This, therefore, is another

ASSUMPTION:

$$V_T \approx U_0 \approx \text{constant}$$

$$W_0 \approx 0 \therefore \alpha_0 \approx 0$$

$$V_0 \approx 0 \therefore \beta_0 \approx 0$$

$$P_0 \approx Q_0 \approx R_0 \approx 0$$

$$(\rho, M, Re, \text{aircraft configuration}) = \text{constant}$$

4.6.4.3.4 Expansion By Taylor Series. As stated earlier, the equations resulting from summing forces and moments are nonlinear and exact solutions are not obtainable. An approximate solution is found by linearizing these equations using a Taylor Series expansion and neglecting higher ordered terms.

As an introduction to this technique, assume some arbitrary non-linear function,  $f(U)$ , having the graphical representation shown in Figure 4.22.

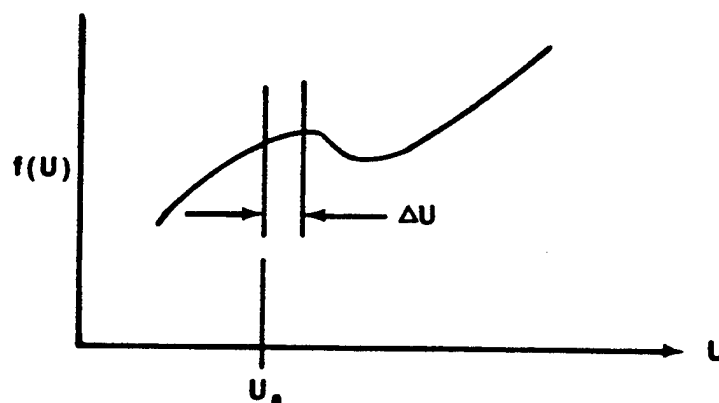


FIGURE 4.22. APPROXIMATION OF AN ARBITRARY FUNCTION BY TAYLOR SERIES

A Taylor Series expansion will approximate the curve over a short span,  $\Delta U$ . The first derivative assumes the function between  $\Delta U$  to be a straight line with slope  $\partial f(U_0)/\partial U$ . This approximation is illustrated in Figure 4.23.

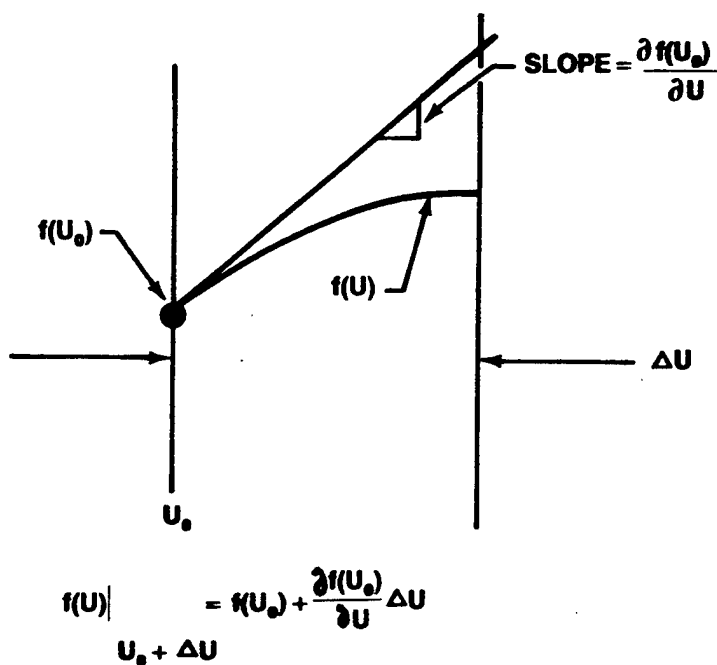
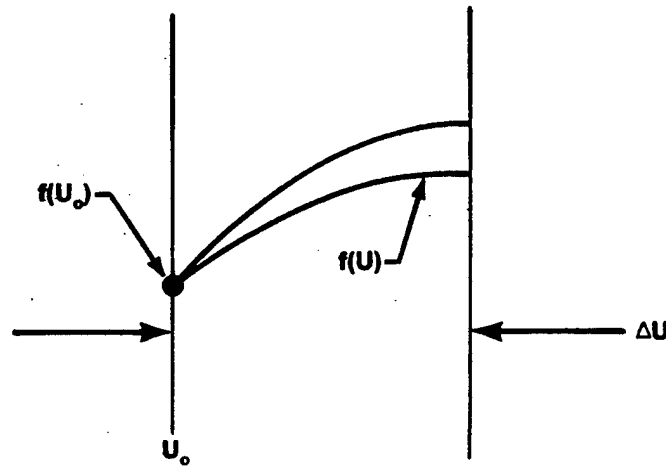


FIGURE 4.23. FIRST ORDER APPROXIMATION BY TAYLOR SERIES

To refine the accuracy of the approximation, a second derivative term is added. The second order approximation is shown in Figure 4.24.



$$f(U) \Big|_{U_0 + \Delta U} = f(U_0) + \frac{\partial f(U_0)}{\partial U} \Delta U + \frac{1}{2!} \frac{\partial^2 f(U_0)}{\partial U^2} (\Delta U)^2$$

FIGURE 4.24. SECOND ORDER APPROXIMATION BY TAYLOR SERIES

Additional accuracy can be obtained by adding higher order derivatives. The resulting Taylor Series expansion has the form

$$\begin{aligned} f(U) \Big|_{U_0 + \Delta U} = & f(U_0) + \frac{\partial f(U_0)}{\partial U} \Delta U + \frac{1}{2!} \frac{\partial^2 f(U_0)}{\partial U^2} (\Delta U)^2 + \frac{1}{3!} \frac{\partial^3 f(U_0)}{\partial U^3} (\Delta U)^3 \\ & + \dots + \frac{1}{n!} \frac{\partial^n f(U_0)}{\partial U^n} (\Delta U)^n \end{aligned} \quad (4.86)$$

If we make  $\Delta U$  smaller, our accuracy will increase and higher order terms can be neglected without significant error. Also since  $\Delta U$  is small,  $(\Delta U)^2$ ,  $(\Delta U)^3$ ,  $(\Delta U)^n$  are very small. Therefore, for small perturbed values of  $U$ , the function can be accurately approximated by

$$f(U) \Big|_{U_0 + \Delta U} = f(U_0) + \frac{\partial f(U_0)}{\partial U} \Delta U \quad (4.87)$$

We can now linearize the aerodynamic forces and moments using this technique. To illustrate, recall the lift term from the longitudinal set of equations ( $L = -F_z$ ). From equation 4.84 we saw that lift was a function of  $U$ ,  $\alpha$ ,  $\dot{\alpha}$ ,  $Q$ ,  $\delta_e$ . The Taylor Series expansion for lift is therefore

$$L = \left[ \begin{array}{l} L_0 + \frac{\partial L}{\partial U} \Delta U + \frac{1}{2} \frac{\partial^2 L}{\partial U^2} \Delta U^2 + \dots \\ + \frac{\partial L}{\partial \alpha} \Delta \alpha + \frac{1}{2} \frac{\partial^2 L}{\partial \alpha^2} \Delta \alpha^2 + \dots \\ + \frac{\partial L}{\partial \dot{\alpha}} \Delta \dot{\alpha} + \dots \\ + \frac{\partial L}{\partial Q} \Delta Q + \dots \\ + \frac{\partial L}{\partial \delta_e} \Delta \delta_e + \dots \end{array} \right] \quad (4.88)$$

where  $L_0 = L(U_0, \alpha_0, \dot{\alpha}_0, Q_0, \delta_{e_0})$

In small perturbation theory, each of the variables is expressed as the sum of an initial value plus a small perturbed value. For example

$$U = U_0 + u, \text{ where } u = \Delta U = U - U_0 \quad (4.89)$$

and

$$\frac{\partial u}{\partial U} = \frac{\partial (U - U_0)}{\partial U} = \frac{\partial U}{\partial U} - \frac{\partial U_0}{\partial U} = 1 \quad (4.90)$$

Therefore

$$\frac{\partial L}{\partial U} = \frac{\partial L}{\partial u} \frac{\partial u}{\partial U} = \frac{\partial L}{\partial u} \quad (4.91)$$

and

$$\Delta U = u$$

The second term of the expression in equation 4.88 then becomes

$$\frac{\partial L}{\partial U} \Delta U = \frac{\partial L}{\partial u} u \quad (4.92)$$

Similarly

$$\frac{\partial L}{\partial Q} \Delta Q = \frac{\partial L}{\partial q} q \quad (4.93)$$

And all other terms follow. We also elect to let  $\alpha = \Delta\alpha$ ,  $\dot{\alpha} = \Delta\dot{\alpha}$  and  $\delta_e = \Delta\delta_e$ . Dropping higher order terms involving  $u^2$ ,  $q^2$ , etc., Equation 4.88 now becomes

$$L = L_0 + \frac{\partial L}{\partial u} u + \frac{\partial L}{\partial \alpha} \alpha + \frac{\partial L}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial L}{\partial q} q + \frac{\partial L}{\partial \delta_e} \delta_e \quad (4.94)$$

Lateral-directional motion is a function of  $\beta$ ,  $\dot{\beta}$ ,  $P$ ,  $R$ ,  $\delta_a$ ,  $\delta_r$  and can be handled in a similar manner. For example, the aerodynamic terms for rolling moment become

$$L_A = L_{A_0} + \frac{\partial L_A}{\partial \beta} \beta + \frac{\partial L_A}{\partial \dot{\beta}} \dot{\beta} + \frac{\partial L_A}{\partial P} P + \frac{\partial L_A}{\partial R} R + \frac{\partial L_A}{\partial \delta_a} \delta_a + \frac{\partial L_A}{\partial \delta_r} \delta_r \quad (4.95)$$

This development can be applied to all of the aerodynamic forces and moments. The equations are linear and account for all variables that have a significant effect on the aerodynamic forces and moments on an aircraft.

The equations resulting from this development can now be substituted into the LHS of the equations of motion.

#### 4.6.5 Direct Thrust Forces and Moments

Since thrust does not always pass through the cg, its effects on both the force and moment equations must be considered (Figure 4.25).

The component of the thrust vector along the x-axis is

$$X_T = T \cos \epsilon \quad (4.96)$$

The component of the thrust vector along the z-axis is

$$Z_T = -T \sin \epsilon \quad (4.97)$$

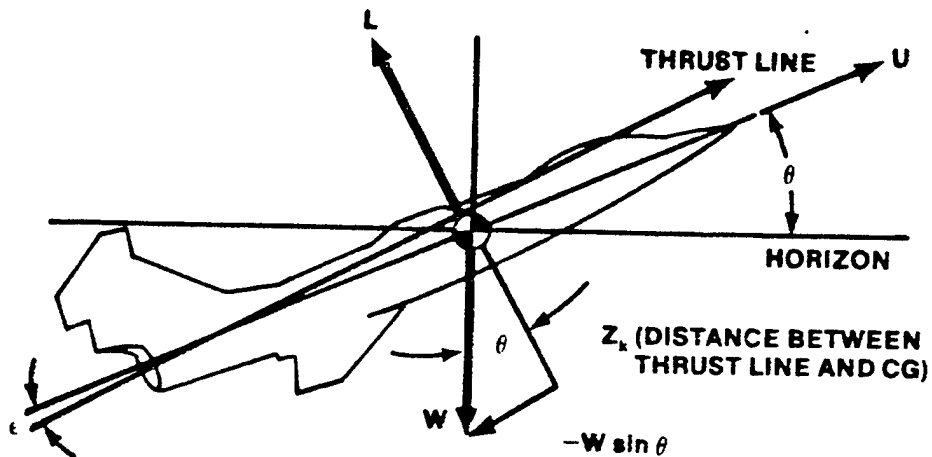


FIGURE 4.25. ORIGIN OF WEIGHT AND THRUST EFFECTS ON FORCES AND MOMENTS

The pitching moment component is,

$$M_T = T (Z_k) = T Z_k \quad (4.98)$$

where  $Z_k$  is the perpendicular distance from the thrust line to the cg and  $\epsilon$  is the thrust angle. For small disturbances, changes in thrust depend only upon the change in forward speed and engine RPM. Therefore, by the same small perturbation analysis used for the aerodynamic forces

$$T = T(U, \delta_{RPM}) \quad (4.99)$$

$$T = T_0 + \frac{\partial T}{\partial U} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} \quad (4.100)$$

Thrust effects will be considered in the longitudinal equations only since the thrust vector is normally in the vertical plane of symmetry and does not affect the lateral-directional motion. When considering engine-out characteristics in multi-engine aircraft, however, the asymmetric thrust effects must be considered. Once again, for clarity,  $X_T$  and  $Z_T$  will be referred to as "drag due to thrust" and "lift due to thrust" ( $\alpha = 0$  assumption in order to use the stability axes). They are components of thrust in the drag (x) and lift (z) directions. Thus:

$$X_T = (T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}) (\cos \epsilon) \quad (4.101)$$

$$Z_T = (T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}) (\sin \epsilon) \quad (4.102)$$

$$M_T = (T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}) (Z_k) \quad (4.103)$$

#### 4.6.6 Gravity Forces

Gravity acts through the cg of an aircraft and, as a result, has no effect on the aircraft moments. It does affect the force equations as shown in Figure 4.25. For longitudinal motion, the only variable to consider is  $\theta$ . For example, consider the effect of weight on the x-axis.

$$X_g = -mg \sin \theta \quad (4.104)$$

Since  $m$  and  $g$  are considered constant,  $\theta$  is the only pertinent variable. Therefore, the expansion of the gravity term,  $X_g$ , can be expressed using the small perturbation assumption as

$$X_g = X_{g_0} + \frac{\partial X_g}{\partial \theta} \theta \quad (X_{g_0} = \text{equilibrium condition of } X_g) \quad (4.105)$$

For simplification, the term  $X_g$  will be referred to as drag due to weight, ( $D_{wt}$ ). This incorporates the small angle assumption that was made in development of the aerodynamic terms; however, the effect is negligible. Therefore, Equation 4.105 becomes

$$D_{wt} = D_{0_{wt}} + \frac{\partial D_{wt}}{\partial \theta} \theta \quad (4.106)$$

Likewise the z-force can be expressed as negative lift due to weight ( $L_{wt}$ ), and the expanded term becomes

$$L_{wt} = L_{0_{wt}} + \frac{\partial L_{wt}}{\partial \theta} \theta \quad (4.107)$$

The effect of gravity on side force depends solely on bank angle ( $\phi$ ), assuming small  $\theta$ . Therefore,

$$Y_{wt} = Y_{0_{wt}} + \frac{\partial Y_{wt}}{\partial \phi} \phi \quad (4.108)$$

These component equations relate the effects of gravity to the equations of motion and can be substituted into the LHS of the equations.

#### 4.6.7 Gyroscopic Moments

Gyroscopic effects are insignificant for most static and dynamic analyses since angular rates are not considered large. They begin to become important as angular rates increase (i.e., P, Q, and R become large). For spin and roll coupling analyses, they are large and gyroscopic effects must be considered. In this basic development of the equations of motion, however, they will be assumed to be negligible.

#### 4.6.8 Expanded LHS Equations

Using the previous developments, the expanded LHS equations become

$$\begin{aligned} \text{"DRAG"} \quad & -[D_0 + \frac{\partial D}{\partial u}u + \frac{\partial D}{\partial \alpha}\alpha + \frac{\partial D}{\partial \dot{\alpha}}\dot{\alpha} + \frac{\partial D}{\partial q}q + \frac{\partial D}{\partial \delta_e}\delta_e] + [T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}] (\cos \epsilon) \\ & - [D_{0_{wt}} + \frac{\partial D}{\partial \theta}\theta] \end{aligned} \quad (4.109)$$

$$\begin{aligned} \text{"LIFT"} \quad & -[L_0 + \frac{\partial L}{\partial u}u + \frac{\partial L}{\partial \alpha}\alpha + \frac{\partial L}{\partial \dot{\alpha}}\dot{\alpha} + \frac{\partial L}{\partial q}q + \frac{\partial L}{\partial \delta_e}\delta_e] + [-T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}] (\sin \epsilon) \\ & + [L_{0_{wt}} + \frac{\partial L}{\partial \theta}\theta] \end{aligned} \quad (4.110)$$

$$\begin{aligned} \text{"PITCH"} \quad & M_{A_0} + \frac{\partial M_A}{\partial u}u + \frac{\partial M_A}{\partial \alpha}\alpha + \frac{\partial M_A}{\partial \dot{\alpha}}\dot{\alpha} + \frac{\partial M_A}{\partial q}q + \frac{\partial M_A}{\partial \delta_e}\delta_e] + [T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}] (Z_k) \end{aligned} \quad (4.111)$$

$$\text{"SIDE"} \quad Y_{A_0} + \frac{\partial Y_A}{\partial \beta}\beta + \frac{\partial Y_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial Y_A}{\partial p}p + \frac{\partial Y_A}{\partial r}r + \frac{\partial Y_A}{\partial \delta_a}\delta_a + \frac{\partial Y_A}{\partial \delta_r}\delta_r + Y_{0_{wt}} + \frac{\partial Y}{\partial \phi}\phi \quad (4.112)$$

$$\text{"ROLL"} \quad L_{A_0} + \frac{\partial L_A}{\partial \beta}\beta + \frac{\partial L_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial L_A}{\partial p}p + \frac{\partial L_A}{\partial r}r + \frac{\partial L_A}{\partial \delta_a}\delta_a + \frac{\partial L_A}{\partial \delta_r}\delta_r \quad (4.113)$$

$$\text{"YAW"} \quad N_{A_0} + \frac{\partial N_A}{\partial \beta}\beta + \frac{\partial N_A}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial N_A}{\partial p}p + \frac{\partial N_A}{\partial r}r + \frac{\partial N_A}{\partial \delta_a}\delta_a + \frac{\partial N_A}{\partial \delta_r}\delta_r \quad (4.114)$$

#### 4.7 RHS IN TERMS OF SMALL PERTURBATIONS

To conform with the Taylor Series expansion of the LHS, the RHS must also be expressed in terms of small perturbations. Recall that each variable is expressed as the sum of an equilibrium value plus a small perturbed value (i.e.,  $U = U_0 + u$ ,  $Q = Q_0 + q$ , etc.). These expressions can be substituted directly into the full set of the RHS equations (Equations 4.23 - 4.25, and 4.59 - 4.61). As an example, the lift equation (z-direction of longitudinal equations) will be expanded. Start with the RHS of Equation 4.25

$$F_z = m (\dot{W} + PV - QU) \quad (4.25)$$

Substitute the initial plus perturbed values for each variable.

$$F_z = m [\dot{W}_0 + \dot{w} + (P_0 + p)(V_0 + v) - (Q_0 + q)(U_0 + u)] \quad (4.116)$$

Multiplying out each term yields:

$$F_z = m [\dot{W}_0 + \dot{w} + P_0 V_0 + p V_0 + P_0 v + pv - Q_0 U_0 - q U_0 - Q_0 u - qu] \quad (4.117)$$

Applying the boundary conditions, (assumptions from subsection 4.6.4.3.3), simplifies the equation to

$$\text{or} \quad F_z = m [\dot{w} + pv - qU_0 - qu] = m [\dot{w} + pv - q(U_0 + u)] \quad (4.118)$$

$$F_z = m [\dot{w} + pv - qU] \quad (4.119)$$

Using this same technique, the set of RHS equations become:

##### Longitudinal

$$\text{"DRAG":} \quad m (\dot{u} + qw - rv) \quad (4.120)$$

$$\text{"LIFT":} \quad m (\dot{w} + pv - qU) \quad (4.121)$$

$$\text{"PITCH":} \quad \dot{q} I_y - pr (I_z - I_x) + (p^2 - r^2) I_{xz} \quad (4.122)$$

### Lateral-Directional

$$\text{"SIDE": } m (\dot{v} + rU - pw) \quad (4.123)$$

$$\text{"ROLL": } \dot{p} I_x + qr (I_z - I_y) - (\dot{r} + pq) I_{xz} \quad (4.124)$$

$$\text{"YAW": } \dot{r} I_z + pq (I_y - I_x) + (qr - \dot{p}) I_{xz} \quad (4.125)$$

Which are valid for small perturbations about an equilibrium condition of steady straight symmetric flight.

## 4.8 REDUCTION OF EQUATIONS TO A USABLE FORM

### 4.8.1 Normalization Of Equations

To put the linearized expressions into a more usable form, each equation is multiplied by a "normalization factor." This factor is different for each equation and is picked to simplify the first term on the RHS of the equation. It is desirable to have the first term of the RHS be either a pure acceleration ( $\dot{u}$ ,  $\dot{p}$ ,  $\dot{q}$ , or  $\dot{r}$ ), or angular rate ( $\dot{\alpha}$ , or  $\dot{\beta}$ ) and these terms were previously identified in equations 4.84 and 4.85 as the longitudinal or lateral-directional variables. As shown previously

$$\alpha \approx \frac{W}{V_T} \quad (4.16)$$

and

$$\beta \approx \frac{V}{V_T} \quad (4.18)$$

Since we have assumed that  $V_T \approx U_0$  and  $V_0 \approx W_0 \approx 0$ , we have

$$\dot{\alpha} \approx \frac{\dot{W}}{U_0} \quad (4.126)$$

and

$$\dot{\beta} \approx \frac{\dot{V}}{U_0} \quad (4.127)$$

Table 4.1 shows the normalizing factors and subsequent equations of motion.

TABLE 4.1  
NORMALIZING FACTORS

Equation	Normalizing Factor	First Term is Now Pure Accel/Ang Rate	Units
"DRAG"	$\frac{1}{m}$	$-\frac{D}{m} + \frac{X_T}{m} + \dots = \dot{u}$	$[\frac{ft}{sec^2}]$ (4.128)
"LIFT"	$\frac{1}{mU_0}$	$-\frac{L}{mU_0} + \frac{Z_T}{mU_0} + \dots = \dot{\alpha}$	$[\frac{rad}{sec}]$ (4.129)
"PITCH"	$\frac{1}{I_y}$	$\frac{M_A}{I_y} + \frac{M_T}{I_y} + \dots = \dot{q}$	$[\frac{rad}{sec^2}]$ (4.130)
"SIDE"	$\frac{1}{mU_0}$	$\frac{Y_A}{mU_0} + \frac{Y_T}{mU_0} + \dots = \dot{\beta}$	$[\frac{rad}{sec}]$ (4.131)
"ROLL"	$\frac{1}{I_x}$	$\frac{L_A}{I_x} + \frac{L_T}{I_x} + \dots = \dot{p}$	$[\frac{rad}{sec^2}]$ (4.132)
"YAW"	$\frac{1}{I_z}$	$\frac{N_A}{I_z} + \frac{N_T}{I_z} + \dots = \dot{r}$	$[\frac{rad}{sec^2}]$ (4.133)

#### 4.8.2 Stability Parameters

Stability parameters are quantities that express the variation of force or moment on the aircraft caused by a disturbance from steady flight. They are simply the partial coefficients ( $\partial L / \partial u$ , etc.) multiplied by their respective normalizing factors. They express the variation of forces or moments caused by a disturbance from steady state. Stability parameters are important because they can be used directly as numerical coefficients in a set of simultaneous differential equations describing the dynamics of an airframe. To demonstrate their development, consider the aerodynamic terms of the lift equation. By multiplying Equation 4.94 by the normalizing factor  $1/mU_0$ , we get

$$\frac{L}{mU_0} = \frac{L_0}{mU_0} + \underbrace{\frac{1}{mU_0} \frac{\partial L}{\partial u}}_{L_u} u + \underbrace{\frac{1}{mU_0} \frac{\partial L}{\partial \alpha}}_{L_\alpha} \alpha + \underbrace{\frac{1}{mU_0} \frac{\partial L}{\partial \dot{\alpha}}}_{L_{\dot{\alpha}}} \dot{\alpha} + \underbrace{\frac{1}{mU_0} \frac{\partial L}{\partial q}}_{L_q} q + \underbrace{\frac{1}{mU_0} \frac{\partial L}{\partial \delta_e}}_{L_{\delta_e}} \delta_e \quad [\frac{rad}{sec}] \quad (4.134)$$

The indicated quantities are defined as stability parameters and the equation becomes

$$\frac{L}{mU_0} = \frac{L_0}{mU_0} + L_u u + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_q q + L_\delta \delta_e \quad \left[ \frac{\text{rad}}{\text{sec}} \right] \quad (4.135)$$

Stability parameters have various dimensions depending on whether they are multiplied by a linear velocity, an angle, or an angular rate. For example,

$$L_u \left[ \frac{1}{\text{ft}} \right] u \left[ \frac{\text{ft}}{\text{sec}} \right] = \left[ \frac{\text{rad}}{\text{sec}} \right]$$

$$L_\alpha \left[ \frac{1}{\text{sec}} \right] \alpha \left[ \text{rad} \right] = \left[ \frac{\text{rad}}{\text{sec}} \right]$$

$$L_{\dot{\alpha}} [\text{none}] \dot{\alpha} \left[ \frac{\text{rad}}{\text{sec}} \right] = \left[ \frac{\text{rad}}{\text{sec}} \right]$$

The lateral-directional motion can be handled in a similar manner. For example, the normalized aerodynamic rolling moment becomes

$$\frac{L_A}{I_x} = \frac{L_{A0}}{I_x} + L_{A_\beta} \beta + L_{A_{\dot{\beta}}} \dot{\beta} + L_{A_p} p + L_{A_r} r + L_{A_{\delta_a}} \delta_a + L_{A_{\delta_r}} \delta_r \quad \left[ \frac{\text{rad}}{\text{sec}^2} \right] \quad (4.136)$$

where

$$L_{A_\beta} = \frac{1}{I_x} \frac{\partial L_A}{\partial \beta} \quad \left[ \frac{1}{\text{sec}^2} \right], \text{ etc.}$$

These stability parameters are sometimes called "dimensional derivatives" or "stability derivative parameters," but we will reserve the word "derivative" to indicate the nondimensional form which can be obtained by rearrangement. This will be developed later in the chapter. See subsection 4.9 for a complete set of equations in stability parameter form.

#### 4.8.3 Simplification Of The Equations

By combining all of the terms derived so far, the resulting equations are somewhat lengthy. In order to economize on effort, several simplifications can be made. For one, all "small effect" terms can be disregarded. Normally these terms are an order of magnitude less than the more predominant terms.

These and other simplifications will help derive a concise and workable set of equations.

#### 4.8.4 Longitudinal Equations

4.8.4.1 Drag Equation. The complete normalized drag equation is

$$\begin{aligned}
 & \underbrace{-\left[ \frac{D_0}{m} + D_{\alpha} \alpha + D_{\dot{\alpha}} \dot{\alpha} + D_u u + D_q q + D_{\delta_e} \delta_e \right]}_{\text{Aero Terms}} - \underbrace{\left[ \frac{D_{0_{wt}}}{m} + D_{\theta} \theta \right]}_{\text{Gravity Terms}} \\
 & + \underbrace{\frac{1}{m} \left[ T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} \right]}_{\text{Thrust Terms}} (\cos \epsilon) = \dot{u} + qw - rv \quad (4.137)
 \end{aligned}$$

Simplifying assumptions

1.  $\frac{T_0}{m} \cos \epsilon - \frac{D_0}{m} - \frac{D_{0_{wt}}}{m} \approx 0$  (Steady State, Sum to Zero)
2.  $\left[ \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} \right] (\cos \epsilon) \approx 0$  (Constant RPM,  $\frac{\partial T}{\partial u}$  is small)
3.  $rv \approx 0$  (No lat-dir motion)  
The small perturbation assumption allows us to analyze the longitudinal motion independent of lateral-directional motion.
4.  $qw \approx 0$  (Order of magnitude)
5.  $D_{\dot{\alpha}}$ , and  $D_q$  are all very small, essentially zero.

The resulting equation is

$$- [D_{\alpha} \alpha + D_u u + D_{\theta} \theta + D_{\delta_e} \delta_e] = \dot{u} \quad (4.138)$$

Rearranging

$$- [\ddot{u} + D_u u + D_\alpha \alpha + D_\theta \theta] = D_{\delta_e} \delta_e \quad (4.139)$$

4.8.4.2 Lift Equation. The complete lift equation is

$$\begin{aligned} & \underbrace{- \left[ \frac{L_0}{mU_0} + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_u u + L_q q + L_{\delta_e} \delta_e \right]}_{\text{Aero Terms}} + \underbrace{\left[ \frac{L_{0_{wt}}}{mU_0} + L_\theta \theta \right]}_{\text{Gravity Terms}} \\ & - \underbrace{\frac{1}{mU_0} \left[ T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} \right]}_{\text{Thrust Terms}} (\sin \epsilon) = \ddot{\alpha} + \frac{pv - qU}{U_0} \end{aligned} \quad (4.140)$$

Simplifying assumptions

1.  $-\frac{L_0}{mU_0} + \frac{L_{0_{wt}}}{mU_0} - \frac{T_0}{mU_0} \sin \epsilon \approx 0$  (Steady State)
2.  $\frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} \approx 0$  (Constant RPM,  $\frac{\partial T}{\partial u}$  is small)
3.  $L_\theta \theta \approx 0$  (Order of magnitude, for small  $\theta$ )
4.  $pv \approx 0$  (No lat-dir motion)
5.  $\frac{qU}{U_0} \approx q$  ( $U \approx U_0$ )

The resulting equation is

$$- [L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_u u + L_q q + L_{\delta_e} \delta_e] = \ddot{\alpha} - q \quad (4.141)$$

Rearranging

$$- L_\alpha \alpha - (1 + L_{\dot{\alpha}}) \dot{\alpha} - L_u u + (1 - L_q) q = L_{\delta_e} \delta_e \quad (4.142)$$

#### 4.8.4.3 Pitch Moment Equation.

$$\begin{aligned} \frac{M_{A_0}}{I_Y} + M_{A_\alpha} \alpha + M_{A_{\dot{\alpha}}} \dot{\alpha} + M_{A_u} u + M_{A_{\delta_e}} \delta_e + M_{A_q} q + \text{Thrust Terms} \\ = \dot{q} - pr \frac{(I_z - I_x)}{I_Y} + \frac{(p^2 - r^2)}{I_Y} I_{xz} \end{aligned} \quad (4.143)$$

This can be simplified as before (using the simplifying assumptions). NOTE: For convenience, we will also drop the "A" subscripts. Thus

$$\dot{q} - M_q q - M_\alpha \alpha - M_{\dot{\alpha}} \dot{\alpha} - M_u u = M_{\delta_e} \delta_e \quad (4.144)$$

We now have three longitudinal equations that are easy to work with. Notice that there are four variables,  $\theta$ ,  $\alpha$ ,  $u$ , and  $q$ , but only three equations. To solve this problem,  $\theta$  can be substituted for  $q$ .

$$q = \dot{\theta}, \text{ and } \dot{q} = \ddot{\theta}$$

This can be verified from the Euler angle transformation for pitch rate where the roll angle,  $\phi$ , is zero.

$$q = \overset{1}{\dot{\theta} \cos \phi} + \overset{0}{\dot{\psi} \sin \phi \cos \theta}$$

$$\therefore q = \dot{\theta}$$

This gives us another restriction to this development of the equations of motion:

ASSUMPTION: The initial bank angle is zero,  $\phi_0 = 0$ .

#### 4.8.5 Lateral-Directional Equations

The complete lateral-directional equations are as follows:

$$\begin{aligned} \frac{Y_{A_0}}{mU_0} + Y_{A_\beta} \beta + Y_{A_{\dot{\beta}}} \dot{\beta} + Y_{A_p} p + Y_{A_r} r + Y_{A_{\delta_a}} \delta_a + Y_{A_{\delta_r}} \delta_r \\ + \frac{Y_{0_{wt}}}{mU_0} + Y_{\phi} \phi = \dot{\beta} + \frac{rU - pw}{U_0} \end{aligned} \quad (4.145)$$

$$\begin{aligned} \frac{L_{A_0}}{I_x} + L_{A_\beta} \beta + L_{A_{\dot{\beta}}} \dot{\beta} + L_{A_p} p + L_{A_r} r + L_{A_{\delta_a}} \delta_a + L_{A_{\delta_r}} \delta_r \\ = \dot{p} + qr \left( \frac{I_z - I_y}{I_x} \right) - (\dot{r} + pq) \frac{I_{xz}}{I_x} \end{aligned} \quad (4.146)$$

$$\begin{aligned} \frac{N_{A_0}}{I_z} + N_{A_\beta} \beta + N_{A_{\dot{\beta}}} \dot{\beta} + N_{A_p} p + N_{A_r} r + N_{A_{\delta_a}} \delta_a + N_{A_{\delta_r}} \delta_r \\ = \dot{r} + pq \left( \frac{I_y - I_x}{I_z} \right) + (qr - \dot{p}) \frac{I_{xz}}{I_z} \end{aligned} \quad (4.147)$$

In order to simplify the equations, the following assumptions are made:

1. A wings level steady state condition exists initially. Therefore,  $L_{A_0}$ ,  $N_{A_0}$ ,  $Y_{A_0}$ , and  $Y_{0_{wt}}$  are zero.
2.  $p \approx \dot{\phi}$ ,  $\dot{p} \approx \ddot{\phi}$  ( $\theta \approx 0$ , see Euler angle transformations for roll rate, equation 4.70)
3. The terms  $Y_{A_{\dot{\beta}}} \dot{\beta}$ ,  $L_{A_{\dot{\beta}}} \dot{\beta}$ , and  $N_{A_{\dot{\beta}}} \dot{\beta}$  are all small, essentially zero.

$$4. \frac{rU}{U_0} \approx r \quad (U \approx U_0)$$

$$5. q \approx 0 \quad (\text{no pitching rate in the Lateral-Direction equations})$$

$$6. pw \approx 0 \quad (\text{no longitudinal motion})$$

NOTE: We will also drop the "A" subscripts for convenience.

#### 4.9 STABILITY PARAMETER FORM OF THE EQUATIONS OF MOTION

Using the previous subsection, the equations of motion reduce to the following stability parameter form:

##### Longitudinal Equations

	( $\theta$ )	( $u$ )	( $\alpha$ )		
"DRAG"	$-D_\theta \theta$	$-\dot{u} - D_u u$	$-D_\alpha \alpha$	$= D_\delta \delta_e$	(4.148)

"LIFT"	$(1-L_q) \dot{\theta}$	$-L_u u$	$-(1+L_{\dot{\alpha}}) \dot{\alpha} - L_\alpha \alpha$	$= L_\delta \delta_e$	(4.149)
--------	------------------------	----------	--	-----------------------	---------

"PITCH"	$\ddot{\theta} - M_q \dot{\theta}$	$-M_u u$	$-M_{\dot{\alpha}} \dot{\alpha} - M_\alpha \alpha$	$= M_\delta \delta_e$	(4.150)
---------	------------------------------------	----------	--	-----------------------	---------

##### Lateral Directional Equations

	( $\beta$ )	( $\phi$ )	( $r$ )		
"SIDE FORCE"	$\dot{\beta} - Y_\beta \beta$	$-Y_p \dot{\phi} - Y_\phi \phi$	$+(1-Y_r) r$	$= Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r$	(4.151)

"ROLLING MOMENT"	$-L_\beta \beta$	$+\ddot{\phi} - L_p \dot{\phi}$	$-\frac{I_{xz}}{I_x} \dot{r} - L_r r$	$= L_{\delta_a} \delta_a + L_{\delta_r} \delta_r$	(4.152)
------------------	------------------	---------------------------------	---------------------------------------	---	---------

"YAWING"	$-N_\beta \beta$	$-\frac{I_{xz}}{I_z} \ddot{\phi} - N_p \dot{\phi}$	$+\dot{r} - N_r r$	$= N_{\delta_a} \delta_a + N_{\delta_r} \delta_r$	(4.153)
----------	------------------	--	--------------------	---	---------

There are six unknowns and six equations. The terms on the RHS are now the inputs or "forcing functions." Therefore, for any input  $\delta_e$ ,  $\delta_a$ , or  $\delta_r$ , the equations can be used solved to get  $\theta$ ,  $u$ ,  $\alpha$ ,  $\beta$ ,  $\phi$ , and  $r$ .

Stability parameter data is usually the format found in textbooks or NASA published aircraft stability and control data reports.

#### 4.10 STABILITY DERIVATIVES

The parametric equations give all the information necessary to describe the motion of any particular aircraft. There is only one problem. When using a wind tunnel model for verification, a scaling factor must be used to find the values for the aircraft. It is difficult, therefore, to compare aircraft using stability parameters. In order to eliminate this requirement, a set of nondimensional equations can be derived. This is best illustrated by an example:

Given the parametric equation for pitching moment,

$$\ddot{\theta} - M_q \dot{\theta} - M_u u - M_\alpha \alpha - M_{\dot{\alpha}} \dot{\alpha} = M_{\delta_e} \delta_e \quad (4.154)$$

Let us derive an equation in which all terms are NONDIMENSIONAL.

The steps in this process are:

1. Take each stability parameter and substitute its coefficient relation and take the derivative at the initial condition, with other variables (i.e.  $u$ ,  $\alpha$ ,  $\dot{\alpha}$ ,  $\delta_e$ ) held constant. From subsonic aerodynamics,

$$M = C_m \frac{1}{2} \rho U^2 S c = M_q q + M_u u + M_{\dot{\alpha}} \dot{\alpha} + M_\alpha \alpha - M_{\delta_e} \delta_e \quad (4.155)$$

Looking at the change in this pitching moment due to a change in pitch rate (as an example) at the initial conditions

$$M_q = \frac{1}{I_y} \frac{\partial M}{\partial q} = \frac{1}{I_y} \frac{\partial (C_m \frac{1}{2} \rho U^2 S c)}{\partial q} \bigg|_0 \quad (4.156)$$

$C_m$  is the only variable that is dependent on  $q$ , therefore,

$$M_q = \frac{\rho U_0^2 S c}{2 I_y} \frac{\partial C_m}{\partial q} \quad (4.157)$$

2. Nondimensionalize the partial term:

$$\frac{\partial C_m}{\partial q} \text{ has dimensions} = \frac{\text{dimensionless}}{\text{rad/sec}} = \text{sec}$$

To nondimensionalize the partial terms, certain compensating factors are customarily used (Table 4.2). In this case, the compensating factor is

$$\frac{c}{2U_0} \frac{[\text{ft}]}{[\text{ft/sec}]} = \text{sec}$$

Multiply and divide Equation (4.157) by the compensating factor and get

$$M_q = \frac{\rho U_0^2 S c}{2 I_y} \left[ \frac{\frac{c}{2U_0}}{\frac{c}{2U_0}} \frac{\partial C_m}{\partial q} \right] \Rightarrow \text{This term is dimensionless} \quad (4.158)$$

Checking

$$\frac{\partial C_m}{\partial (\frac{cq}{2U_0})} = \frac{\text{dimensionless}}{\frac{\text{ft/sec}}{\text{ft/sec}}} = \text{dimensionless}$$

This is called a stability derivative and is written

$$C_{m_q} = \frac{\partial C_m}{\partial (\frac{cq}{2U_0})}$$

The basic nondimensional form  $C_{m_q}$  is important because correlation between geometrically similar airframes or the same airframe at different flight conditions is easily attained with stability derivatives (this cannot easily be done with stability parameters). Additionally, aerodynamic stability derivative data from wind tunnel tests, flight tests, and theoretical analyses are usually presented in nondimensional form.

Stability derivatives generally fall into two classes: static and dynamic. Static derivatives arise from the position of the airframe with

respect to the relative wind (i.e.,  $C_L$ ,  $C_m$ ,  $C_n$ ,  $C_l$ ). Whereas, dynamic derivatives arise from the motion (velocities) of the airframe i.e.,  $C_{L_\alpha}$ ,  $C_{m_u}$ ,  $C_{n_r}$ ,  $C_{l_p}$ .

3. The entire term with stability derivative is

$$M_q = \frac{\rho U_0^2 Sc}{2I_y} \frac{C}{2U_0} C_{m_q} q \quad (4.160)$$

4. We can do the same for each term in the parametric equation. For example,

$$M_u = \frac{1}{I_y} \frac{\partial M}{\partial u} = \frac{1}{I_y} \left. \frac{\partial (C_m \frac{1}{2} \rho U^2 Sc)}{\partial u} \right|_0 \quad (4.161)$$

Since both  $C_m$  and  $U$  are functions of  $u$ , then

$$M_u = \frac{\rho Sc}{2I_y} \left. \frac{\partial (C_m U^2)}{\partial u} \right|_0 \quad (4.162)$$

$$M_u = \frac{\rho Sc}{2I_y} [U_0^2 \frac{\partial C_m}{\partial u} + 2C_{m_0} U_0] \quad (4.163)$$

$$\therefore M_u = \frac{\rho U_0^2 Sc}{2I_y} \left[ \frac{\partial C_m}{\partial u} + \frac{2C_{m_0}}{U_0} \right] \quad (4.164)$$

but  $C_{m_0} = 0$  since the initial conditions are steady state. The compensating factor for this case is  $1/U_0$

$$\therefore M_u = \frac{\rho U_0^2 Sc}{2I_y} \frac{1}{U_0} \left[ \frac{\partial C_m}{\partial (\frac{u}{U_0})} \right] u \quad (4.165)$$

$C_{m_u}$

5. Once all of the terms have been derived, they are substituted into the original equation, and multiplied by

$$\frac{2I_y}{\rho U_0^2 S c}$$

which gives

$$\frac{2I_y}{\rho U_0^2 S c} \ddot{\theta} - \frac{c}{2U_0} C_{m_q} \dot{\theta} - \frac{1}{U_0} C_{m_u} u - \frac{c}{2U_0} C_{m_{\dot{\alpha}}} \dot{\alpha} - C_{m_{\alpha}} \alpha = C_{m_{\delta_e}} \delta_e \quad (4.167)$$

The compensating factors for all of the variables are listed in Table 4.2.

TABLE 4.2  
COMPENSATING FACTORS  
Compensating  
Factor

<u>Variable</u>	<u>Compensating Factor</u>	<u>Nondimensional Variable</u>
p (rad/sec)	$\frac{b}{2U_0}$	$\frac{bp}{2U_0} = \frac{b\dot{\phi}}{2U_0}$
q (rad/sec)	$\frac{c}{2U_0}$	$\frac{cq}{2U_0} = \frac{c\dot{\theta}}{2U_0}$
r (rad/sec)	$\frac{b}{2U_0}$	$\frac{br}{2U_0}$
$\dot{\beta}$ (rad/sec)	$\frac{b}{2U_0}$	$\frac{b\dot{\beta}}{2U_0}$
$\dot{\alpha}$ (rad/sec)	$\frac{c}{2U_0}$	$\frac{c\dot{\alpha}}{2U_0}$
u (ft/sec)	$\frac{1}{U_0}$	$\frac{u}{U_0}$
$\alpha$ (radians)	none	$\alpha$
$\beta$ (radians)	none	$\beta$

#### 4.10 STABILITY DERIVATIVE FORM OF THE EQUATIONS OF MOTION

The simplified equations of motion in stability derivative form are shown in Table 4.3. The derivation of these equations has been presented to give an understanding of their origin and what they represent. It is not necessary to

TABLE 4.3

## LONGITUDINAL EQUATIONS

	( $\theta$ )	( $u$ )	( $\alpha$ )	(forcing functions)
"DRAG"	$-C_{D_\theta}$	$-\frac{2m}{\rho U_0^2 S} \dot{u} - \frac{1}{U_0} (C_{D_u} + 2C_{D_0})u$	$-C_{D_\alpha} \alpha$	$= C_{D_{\delta_\theta}} \delta_\theta$ (4.168)
"LIFT"	$\left( \frac{2m}{\rho S U_0} - \frac{C}{2U_0} C_{L_q} \right) \dot{\theta}$	$-\frac{1}{U_0} (C_{L_u} - 2C_{L_0})u - \left( \frac{2m}{\rho S U_0} + \frac{C}{2U_0} C_{L_\alpha} \right) \dot{\alpha} - C_{L_\alpha} \alpha$	$= C_{L_{\delta_\theta}} \delta_\theta$	(4.169)
"PITCH MOMENT"	$\frac{2I_y}{\rho S U_0^2 c} \ddot{\theta} - \frac{C}{2U_0} C_{m_q} \dot{\theta}$	$-\frac{1}{U_0} C_{m_u} u$	$-\frac{C}{2U_0} C_{m_\alpha} \dot{\alpha} - C_{m_\alpha} \alpha$	$= C_{m_{\delta_\theta}} \delta_\theta$ (4.170)

4.55

## LATERAL-DIRECTIONAL EQUATIONS

	( $\beta$ )	( $\phi$ )	( $r$ )	(forcing functions)
"SIDE FORCE"	$\frac{2m}{\rho S U_0} \dot{\beta} - C_{Y_\beta} \beta - \frac{b}{2U_0} C_{Y_p} \dot{\phi} - C_{Y_r} \phi$	$+\left( \frac{2m}{\rho S U_0} - \frac{b}{2U_0} C_{Y_r} \right) r$	$= C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r$	(4.171)
"ROLLING MOMENT"	$-C_{l_\beta} \beta + \frac{2I_x}{\rho S b U_0^2} \dot{\phi} - \frac{b}{2U_0} C_{l_p} \dot{\phi}$	$-\frac{2I_{xz}}{\rho S b U_0^2} r - \frac{b}{2U_0} C_{l_r} r$	$= C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r$	(4.172)
"YAWING MOMENT"	$-C_{n_\beta} \beta - \frac{2I_{xz}}{\rho S b U_0^2} \dot{\phi} - \frac{b}{2U_0} C_{n_p} \dot{\phi}$	$-\frac{2I_z}{\rho S b U_0^2} r - \frac{b}{2U_0} C_{n_r} r$	$= C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r$	(4.173)

be able to derive each and every one of the equations. It is important, however, to understand several facts about the nondimensional equations.

1. Since these equations are nondimensional, they can be used to compare aircraft characteristics of geometrically similar airframes.
2. Stability derivatives can be thought of as if they were stability parameters. Therefore,  $C_{m\alpha}$  refers to the same aerodynamic characteristics as  $M_\alpha$ , only it is in a nondimensional form.
3. Most aircraft designers and builders are accustomed to speaking in terms of stability derivatives. Therefore, it is a good idea to develop a "feel" for all of the important ones.
4. These equations as well as the parametric equations (eq. 4.70 to 4.72) describe the complete motion of an aircraft. They can be programmed directly into a computer and connected to a flight simulator. They may also be used in cursory design analyses. Due to their simplicity, they are especially useful as an analytical tool to investigate aircraft handling qualities and determine the effect of changes in aircraft design.
5. Sometimes non-dimensional stability derivative data are tabulated as a function of  $\alpha$  and Mach number to cover the entire aircraft flight envelope. Equations of motion for other than 1 g flight can be obtained from this format given  $\alpha$  corresponding to a particular load factor.

#### 4.11 AIRCRAFT TRANSFER FUNCTIONS

If the Laplace transform of the stability parameter equations of motion (subsection 4.9) are taken, assuming zero initial conditions, and the equations are written in matrix notation, the following equations result (in terms of stability parameters for the aircraft stability axis system):

Longitudinal Axis:

$$\begin{bmatrix} -D_\theta & -(s + D_u) & -D_\alpha \\ (1-L_q)s & -L_u & -(1+L_\alpha)s - L_\alpha \\ s^2 - M_q s & -M_u & -M_\alpha s - M_\alpha \end{bmatrix} \begin{bmatrix} \theta \\ u \\ \alpha \end{bmatrix} = \begin{bmatrix} D_{\delta_e} \\ L_{\delta_e} \\ M_{\delta_e} \end{bmatrix} [\delta_e] \quad (4.174)$$

Lateral-Directional Axes:

$$\begin{bmatrix} s - Y_p & -(Y_p s + Y_\phi) & (1 - Y_r) \\ -L_\beta & s^2 - L_p s & -\frac{I_{xz}}{I_x} s - L_r \\ -N_\beta & -\frac{I_{xz}}{I_z} s^2 - N_p s & s - N_r \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ r \end{bmatrix} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \quad (4.175)$$

The above matrix equations are all that are needed for transfer function derivation. As an example, the transfer function relating  $\alpha$  to  $\delta_e$  is

$$G_{\delta_e}^\alpha(s) = \frac{\alpha(s)}{\delta_e(s)} \quad (4.176)$$

which can be found from Cramer's rule as

$$G_{\delta_e}^\alpha(s) = \frac{\begin{vmatrix} -D_\theta & -(s + D_u) & D_{\delta_e} \\ (1-L_q)s & -L_u & L_{\delta_e} \\ s^2 - M_q s & -M_u & M_{\delta_e} \end{vmatrix}}{\begin{vmatrix} -D_\theta & -(s + D_u) & -D_\alpha \\ (1-L_q)s & -L_u & -(1+L_\alpha)s - L_\alpha \\ s^2 - M_q s & -M_u & -M_\alpha s - M_\alpha \end{vmatrix}} \quad (4.177)$$

where the determinant in the denominator yields the characteristic equation of the unaugmented aircraft. Similar derivations can be used to find any single input, single output transfer function for the longitudinal or lateral-directional axes.

#### 4.12 AIRCRAFT TRANSFER FUNCTION DATA PRESENTATION

Examples of three formats used to present aircraft transfer function data are shown in Figure 4.26 and Tables 4.4 and 4.5 for the lateral-directional axes of the A-7A.

The data presented in Figure 4.26 can be used in the non-dimensional stability derivative equations. The data presented in Table 4.4 can be used in the dimensional stability parameter equations. Both approaches yield the same transfer functions. The data in Table 4.5 are presented in transfer function format already (body axis system in this case). The symbol  $\Delta$  is the lateral-directional characteristic equation (denominator term). The numerator terms are presented as

$$N_{\text{input}}^{\text{output}}(s)$$

for example

$$N_{\delta_a}^p(s)$$

is the numerator term for roll rate due to aileron deflection.  $1/T$  denotes real axis poles or zeros (discussed in Chapter 13).  $\zeta$  and  $\omega$  denotes damping ratio and natural frequency of poles or zeros (regardless of the subscript in the table). These must be converted to the form

$$s = -\sigma \pm \omega_d j \quad (4.178)$$

where

$$\sigma = \zeta\omega_n \quad \text{and} \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (4.179)$$

$A$  is the root locus gain (discussed in Chapter 13) of the transfer function. For example, the transfer function

$$G_{\delta_a}^p(s) = \frac{N_{\delta_a}^p(s)}{\Delta(s)} \quad (4.180)$$

at 0.6 Mach, sea level has the characteristic equation

$$\Delta(s) = (s + 0.0411)(s + 4.46) [s^2 + 2(0.202)(2.91)s + (2.91)^2] \quad (4.181)$$

or

$$\Delta(s) = (s + 0.0411)(s + 4.46) [s + 0.59 \pm 2.85j] \quad (4.182)$$

and

$$N_{\delta_a}^p(s) = 28.4(s - 0.00234)(s^2 + 2(0.217)(3.05)s + (3.05)^2) \quad (4.183)$$

The units are radians for  $\delta_a$  and radians per second for  $p$ , or degrees for  $\delta_a$  and degrees per second for  $p$ , since both the input and output parameters have similar units. If the input and output parameters have different units, as is the case in the transfer function

$$N_{\delta_a}^{a_y}(s)$$

then the units are feet per second squared for  $a_y$  and radians for  $\delta_a$ . If in doubt, always assume radians for angular parameters.

Tables 4.6 and 4.7 give typical values for stability derivatives. Note that the tables use the general body axes (not the stability axes) which use the forces  $\bar{X}$  and  $\bar{Z}$ , not  $\bar{L}$  and  $\bar{D}$ .

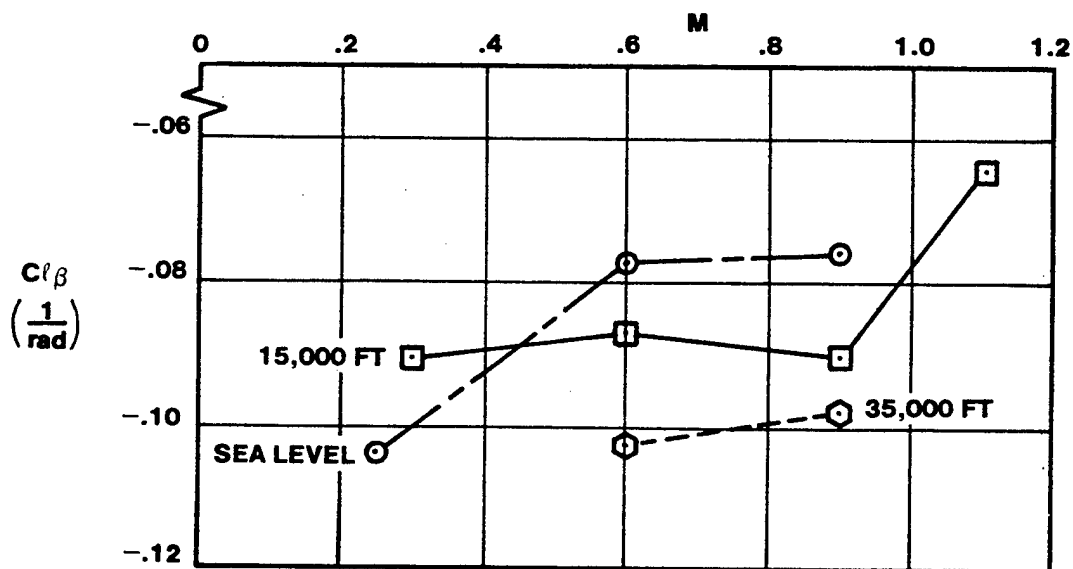
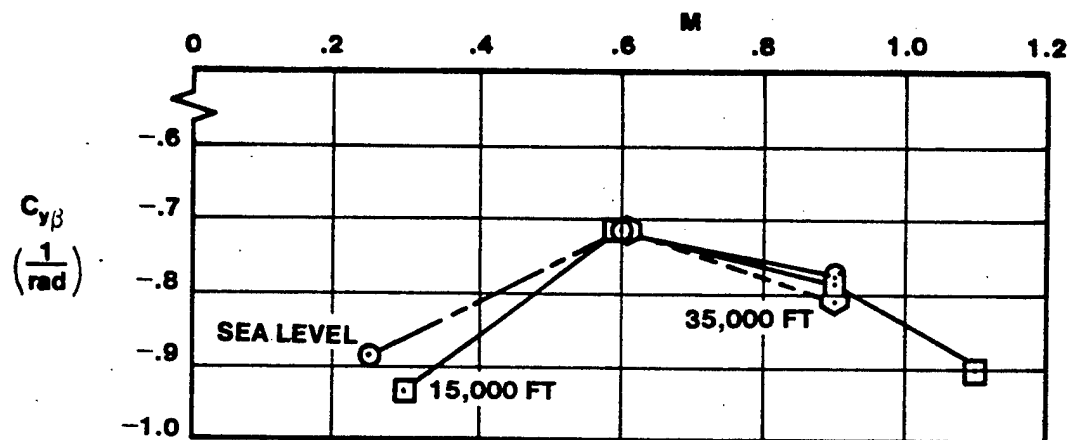


FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA

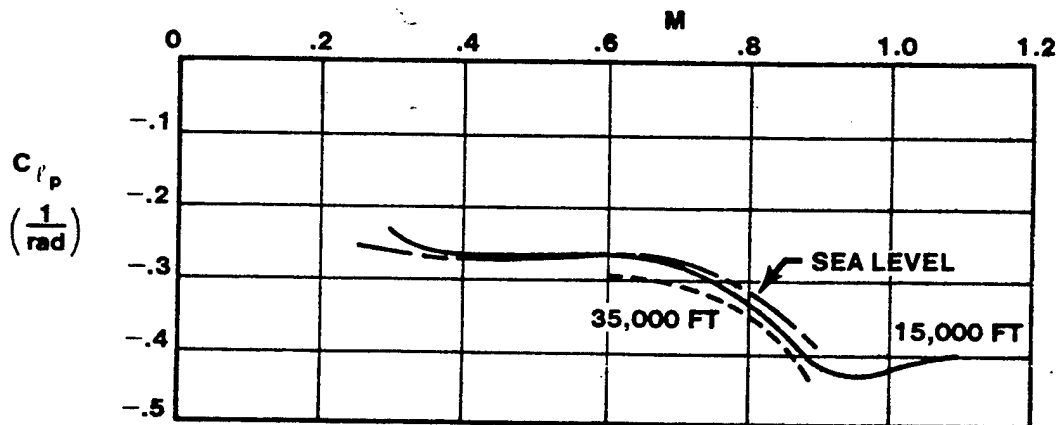
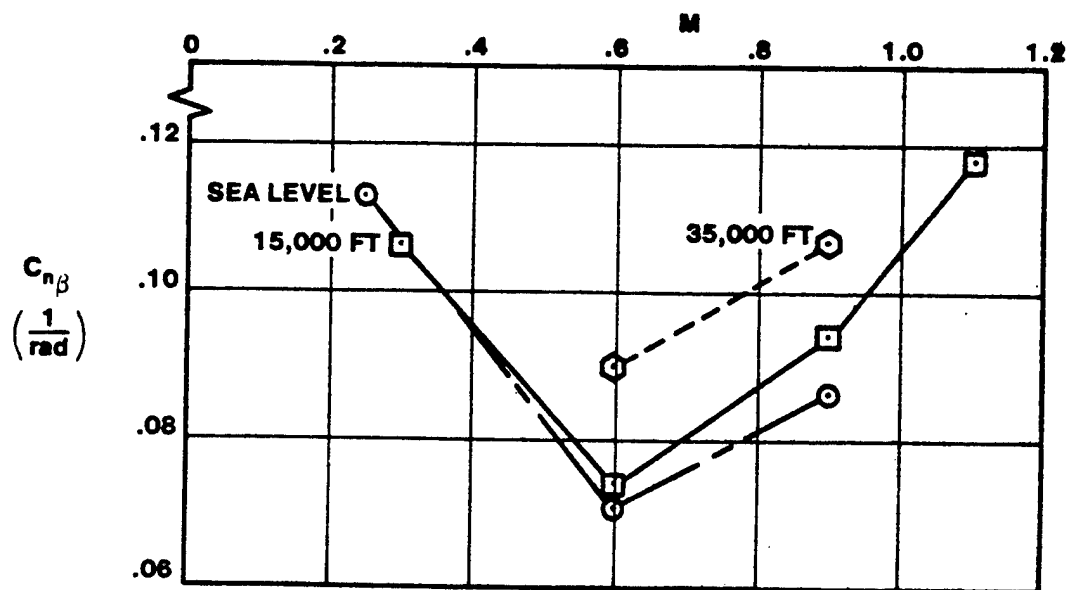


FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA (continued)

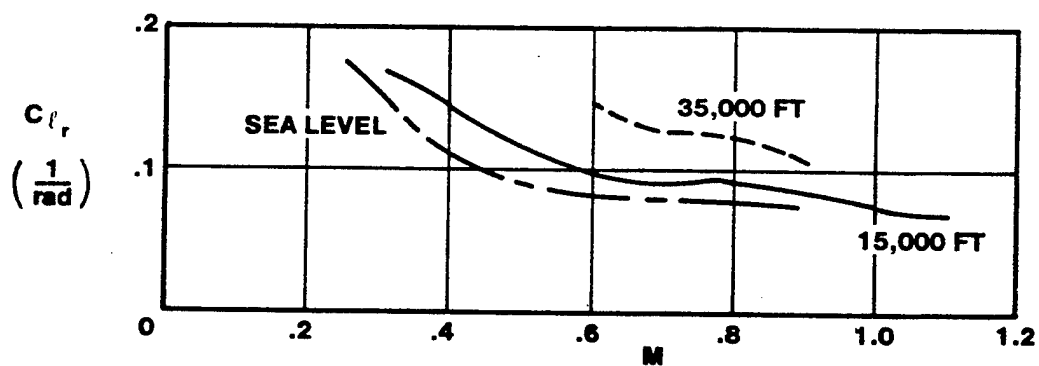
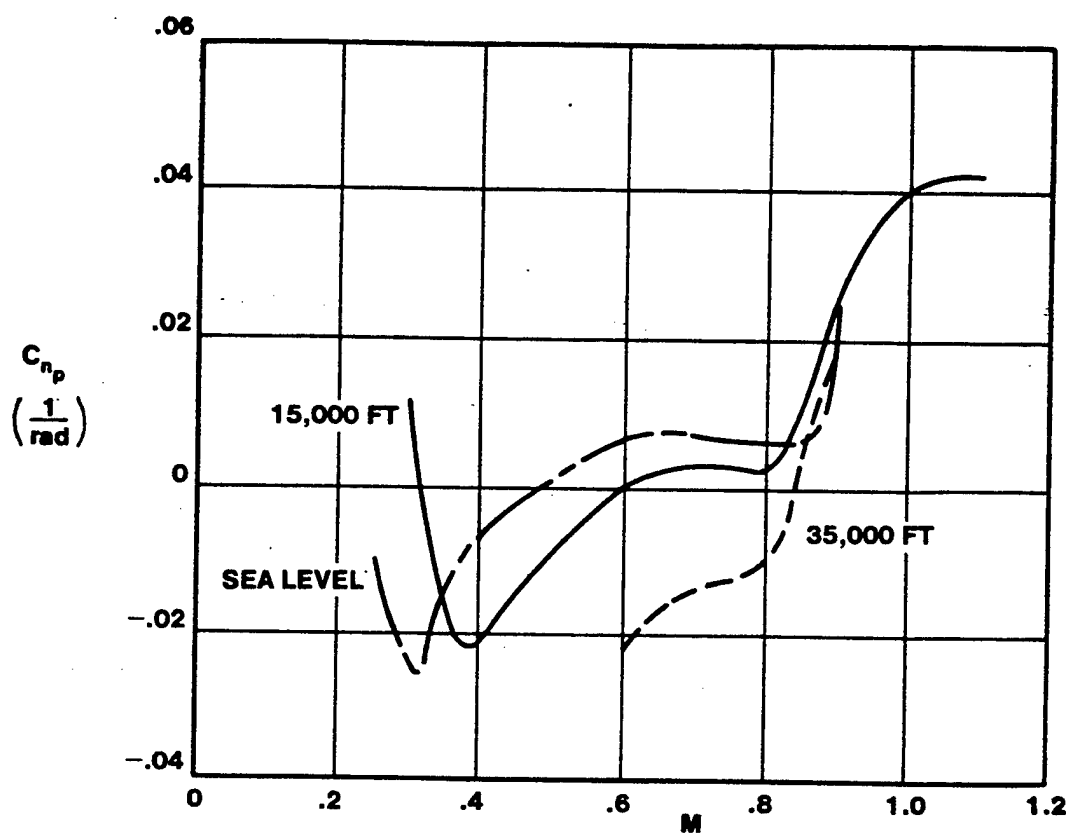
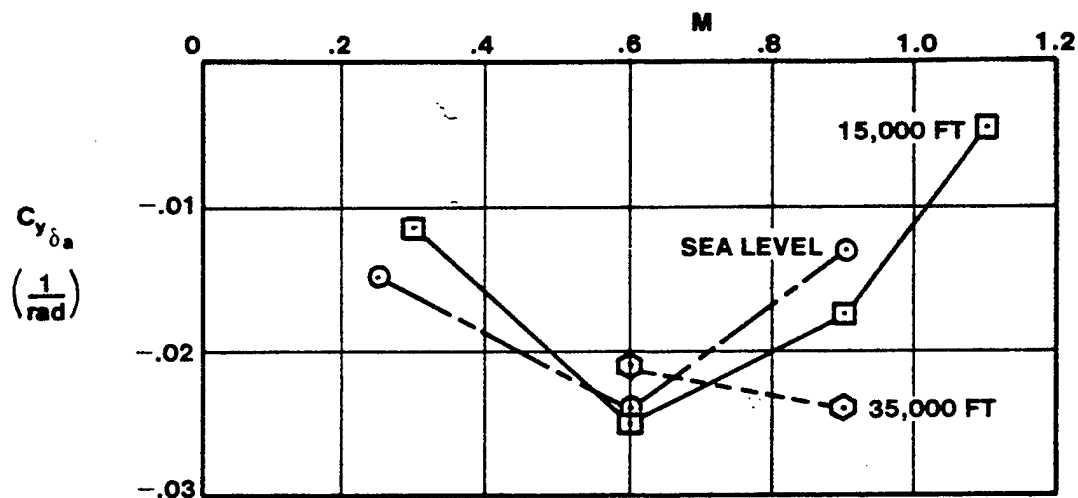
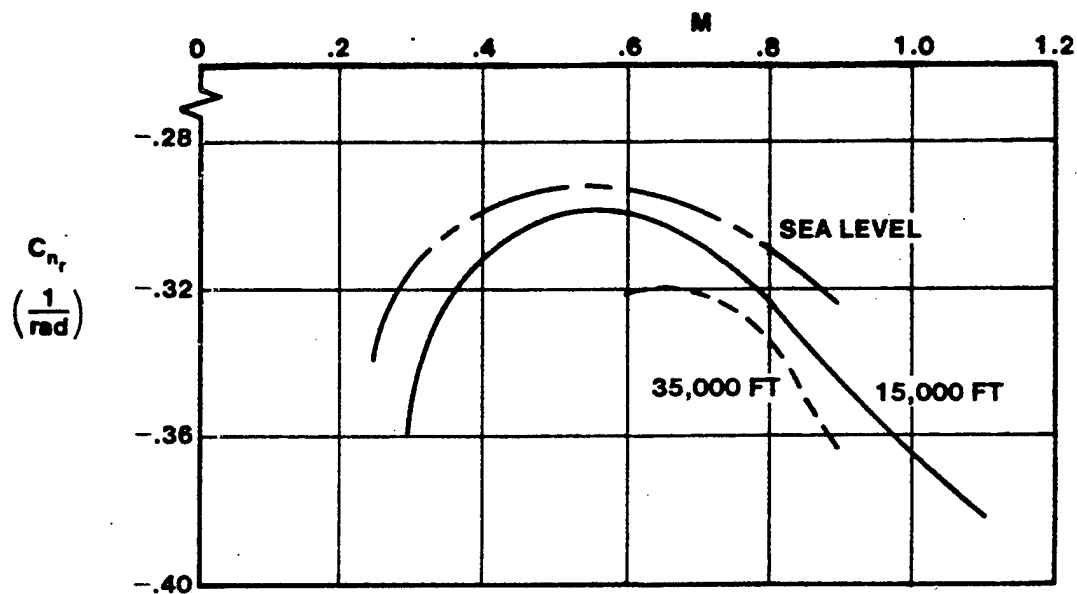


FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA (continued)



$$\delta_a = \frac{\delta_{aR} - \delta_{aL}}{2} \quad (\text{INCLUDES SPOILER EFFECTS})$$

FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA (continued)

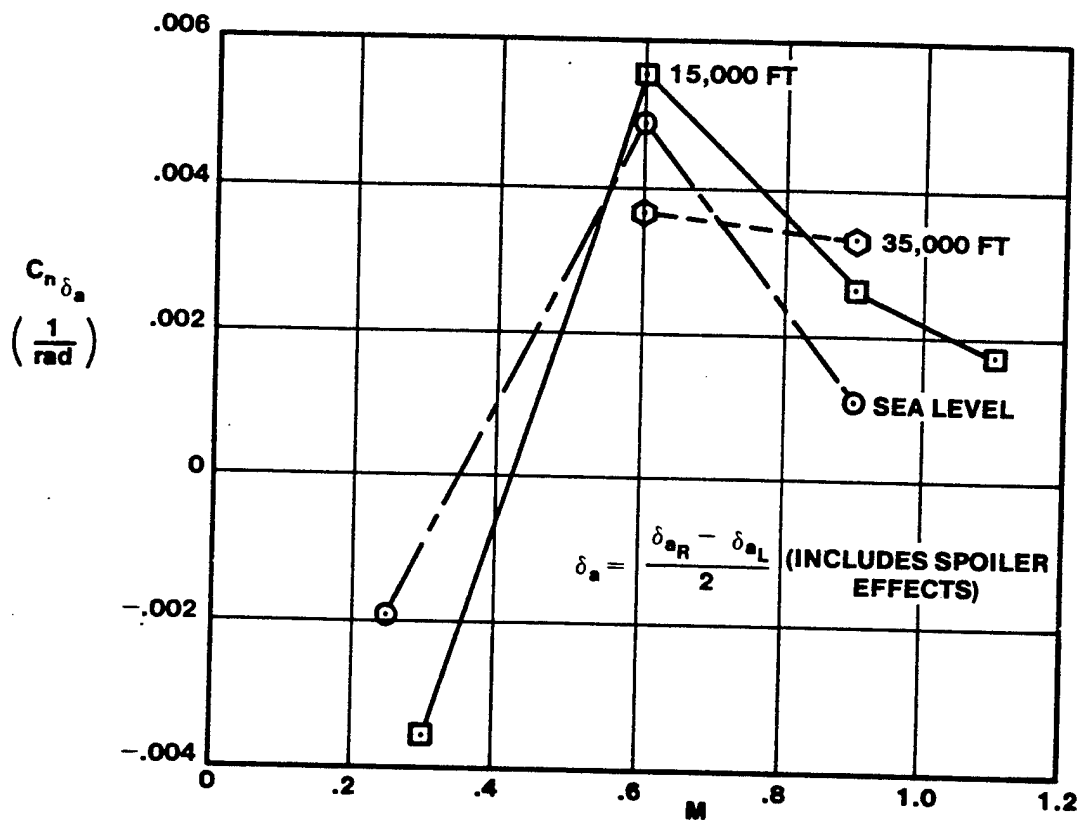
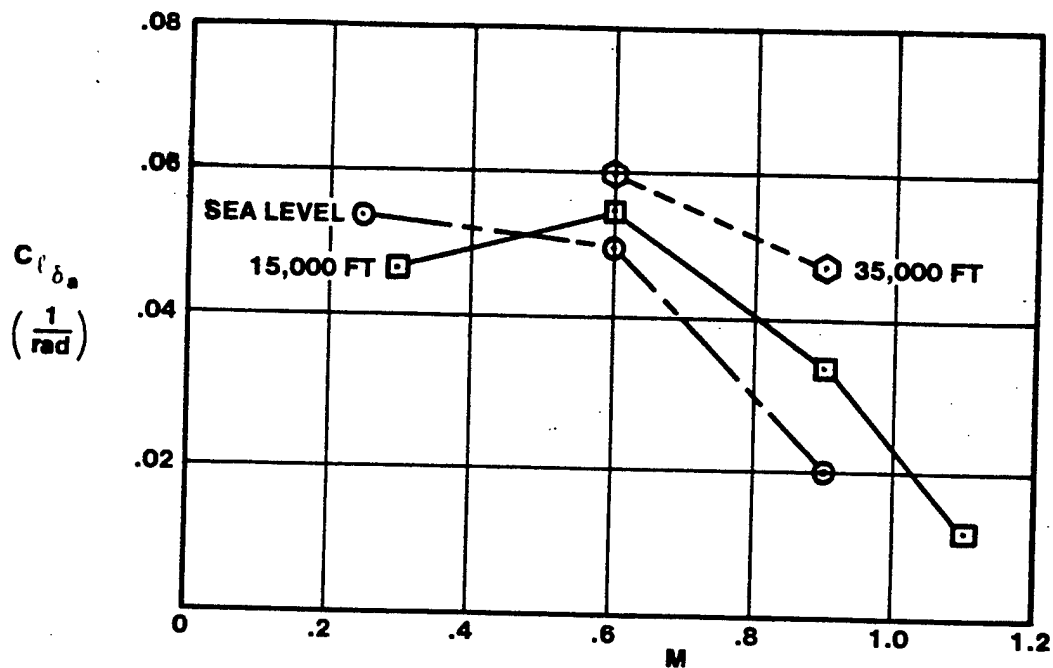


FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA (continued)

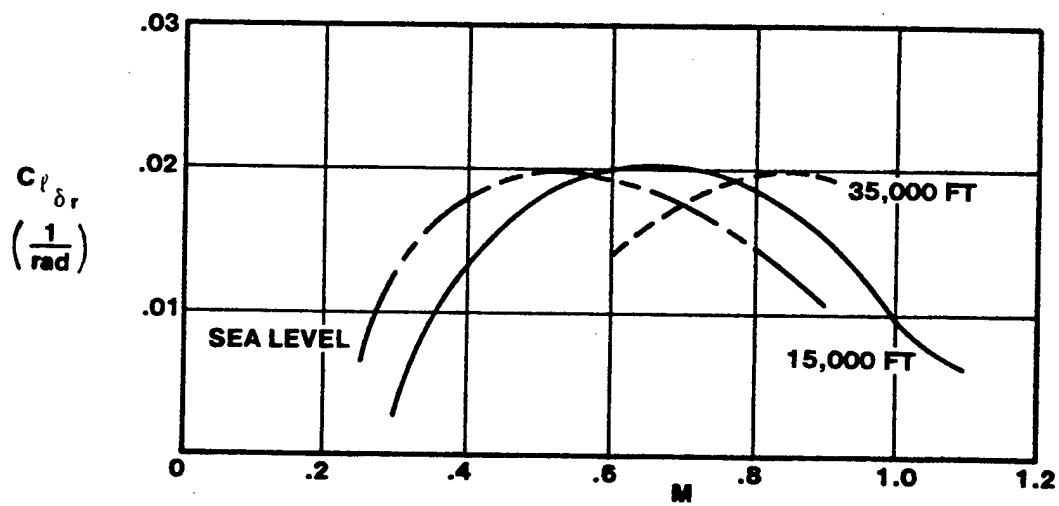
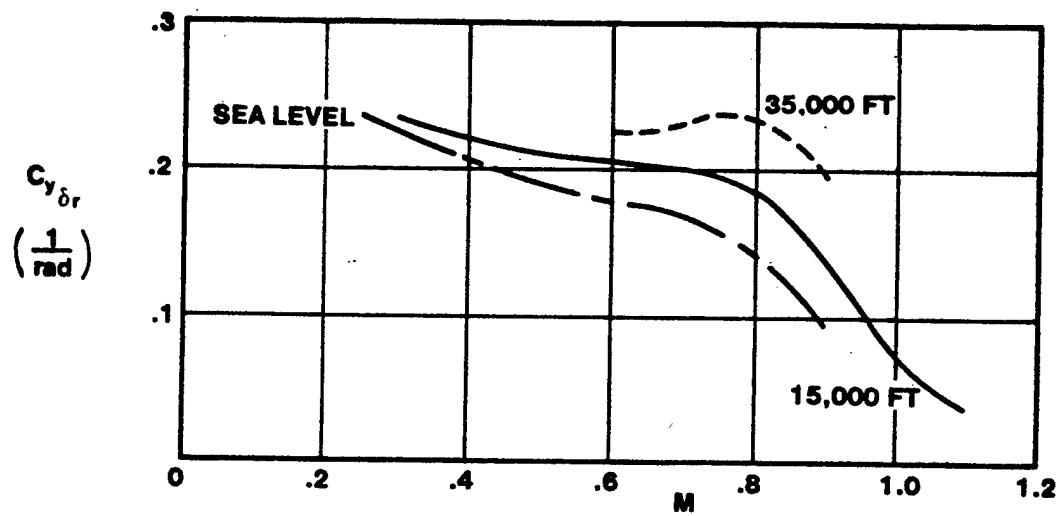


FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA (continued)

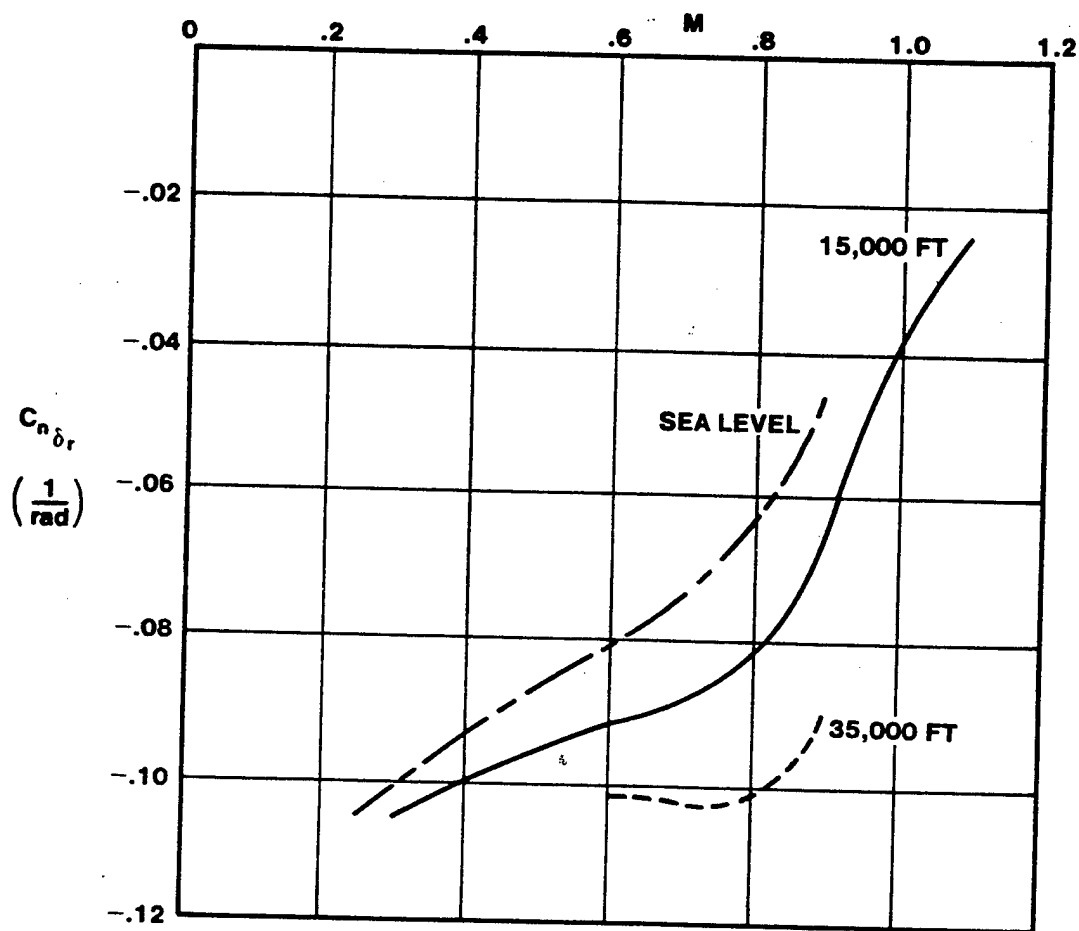


FIGURE 4.26 TYPICAL STABILITY DERIVATIVE DATA (concluded)

TABLE 4.4  
LATERAL-DIRECTIONAL STABILITY PARAMETERS FOR THE A-7A

Note: Data are for body-fixed centerline axes, clean flexible airplane

	FLIGHT CONDITION								
	1	2	3	4	5	6	7	8	9
h	0	0	0	15,000	15,000	15,000	15,000	35,000	35,000
M	0.25	0.6	0.9	0.3	0.6	0.9	1.1	0.6	0.9
$Y_v^*$	-0.162	-0.314	-0.514	-0.122	-0.187	-0.310	-0.435	-0.0847	-0.145
$Y_{\delta_a}$	-0.00274	-0.0105	-0.00857	-0.00150	-0.00655	-0.00691	-0.00216	-0.00267	-0.00427
$Y_{\delta_r}$	0.0430	0.0769	0.0626	0.0307	0.0537	0.0550	0.0192	0.0267	0.0347
$L_{\beta}^*$	-11.9	-44.8	-98.0	-8.79	-29.2	-66.0	-71.2	-14.9	-30.6
$L_p$	-2.00	-4.46	-9.75	-1.38	-2.73	-6.19	-7.31	-1.40	-3.00
$L_r$	1.18	1.15	1.38	0.857	0.868	0.843	0.859	0.599	0.563
$L'_{\delta_a}$	5.34	28.4	25.2	3.75 <sup>b</sup>	17.6	24.1	12.5	7.96	14.2
$L'_{\delta_r}$	2.22	11.4	13.2	1.82	7.27	11.2	7.27	3.09	6.55
$N_{\beta}^*$	1.28	5.74	17.2	0.948	3.12	10.2	21.9	1.38	4.72
$N_p$	-0.0870	-0.168	-0.319	-0.0310	-0.116	-0.207	-0.169	-0.0799	-0.112
$N_r$	-0.369	-0.905	-1.54	-0.271	-0.541	-0.975	-1.33	-0.247	-0.455
$N_{\delta_a}$	0.402	2.08	1.56	0.280	1.37	1.64	1.04	0.652	1.01
$N_{\delta_r}$	-1.93	-8.61	-11.1	-1.56	-5.54	-8.80	-4.83	-2.54	-5.11

$*\beta = V/V_T$

TABLE 4.5  
AILERON LATERAL TRANSFER FUNCTION FACTORS FOR THE A-7A

Note: Data are for body-fixed centerline axes, clean flexible airplane

		FLIGHT CONDITION								
		1	2	3	4	5	6	7	8	9
h		0	0	0	15,000	15,000	15,000	15,000	35,000	35,000
M		0.25	0.6	0.9	0.3	0.6	0.9	1.1	0.6	0.9
$\Delta$	$1/T_s$	0.0462	0.0411	0.0180	0.0449	0.0435	0.0214	0.0102	0.0319	0.0191
	$1/T_R$	1.62	4.46	9.75	0.968	2.71	6.17	7.15	1.28	2.92
	$\zeta_d$	0.237	0.202	0.218	0.231	0.156	0.175	0.189	0.114	0.128
	$\omega_d$	1.81	2.91	4.68	1.65	2.29	3.66	5.03	1.81	2.58
$N_{\delta_a}^p$	$A_p$	5.34	28.4	25.2	3.75	17.6	24.1	12.5	7.96	14.2
	$1/T_{p1}$	-0.0219	-0.00234	-0.00113	-0.0232	-0.00347	-0.00144	-0.00137	-0.00718	-0.00241
	$\zeta_p$	0.217	0.217	0.222	0.191	0.173	0.176	0.173	0.122	0.124
	$\omega_p$	1.49	3.05	4.91	1.27	2.34	3.87	5.33	1.62	2.64
$N_{\delta_a}^\phi$	$A_\phi$	5.42	28.5	25.2	3.81	17.7	24.1	12.5	8.04	14.3
	$\zeta_\phi$	0.210	0.217	0.222	0.183	0.173	0.177	0.175	0.119	0.124
	$\omega_\phi$	1.51	3.05	4.91	1.29	2.34	3.87	5.32	1.62	2.64
$N_{\delta_a}^r$	$A_r$	0.402	2.08	1.56	0.280	1.37	1.64	1.04	0.652	1.01
	$1/T_{r1}$	0.596	1.12	1.13	0.445	0.777	0.944	0.581	0.420	0.593
	$\zeta_r$	0.0852	0.287	0.597	0.146	0.151	0.446	0.638	0.0198	0.193
	$\omega_r$	2.35	2.29	3.26	2.18	2.13	2.78	3.98	2.03	2.45
$N_{\delta_a}^\beta$	$A_\beta$	-0.00274	-0.0105	-0.00857	-0.00150	-0.00655	-0.00691	-0.00216	-0.00267	-0.00427
	$1/T_{\beta_1} (\zeta_\beta)$	(0.885)	3.26	7.76	(0.726)	2.21	5.77	10.7	0.793	(0.872)
	$1/T_{\beta_2} (\omega_\beta)$	(0.667)	-0.627	-0.254	(0.471)	-1.63	-0.245	-0.113	-0.422	(10.6)
	$1/T_{\beta_3}$	-233	63.1	78.2	-391	23.2	86.8	188	-147	-0.545
$N_{\delta_a}^{s_y}$ CG	$A_{s_y}$	-0.766	-7.06	-8.61	-0.477	-4.16	-6.58	-2.51	-1.56	-0.0374
	$1/T_{s_y1} (\zeta_{s_y2})$	(0.943)	2.29	-1.16	(0.758)	1.32	-0.596	-0.146	0.290	(0.801)
	$1/T_{s_y2} (\omega_{s_y2})$	(0.648)	5.92	-1.84	(0.461)	3.12	-2.66	-7.93	0.961	(2.34)
	$\zeta_{s_y} (1/T_{s_y3})$	0.0896	-0.810	(3.65)	0.0673	-0.294	(3.79)	0.897	0.0499	-0.113
	$\omega_{s_y} (1/T_{s_y4})$	6.37	1.76	(10.7)	7.10	1.99	(-6.63)	9.31	3.92	1.30

TABLE 4.5 (concluded)  
AILERON LATERAL TRANSFER FUNCTION FACTORS FOR THE A-7A

Note: Data are for body-fixed centerline axes, clean flexible airplane

		FLIGHT CONDITION								
		1	2	3	4	5	6	7	8	9
h		0	0	0	15,000	15,000	15,000	15,000	35,000	35,000
M		0.25	0.6	0.9	0.3	0.6	0.9	1.1	0.6	0.9
$\Delta$	$1/T_s$	0.0462	0.0411	0.0180	0.0449	0.0435	0.0214	0.0102	0.0319	0.0191
	$1/T_R$	1.62	4.46	9.75	0.968	2.71	6.17	7.15	1.28	2.92
	$\zeta_d$	0.237	0.202	0.218	0.231	0.156	0.175	0.189	0.114	0.128
	$\omega_d$	1.81	2.91	4.68	1.65	2.29	3.66	5.03	1.81	2.58
$N_{\delta_r}^p$	$A_p$	2.22	11.4	13.2	18.2	7.27	11.2	7.27	3.09	6.55
	$1/T_{p_1}$	-0.0224	-0.00242	-0.00117	-0.0237	-0.00352	-0.00147	-0.00141	-0.00723	-0.00243
	$1/T_{p_2}$	2.68	5.35	8.31	2.33	4.31	6.63	5.56	3.16	4.39
	$1/T_{p_3}$	-3.38	-5.31	-7.88	-2.79	-4.45	-6.33	-4.55	-3.44	-4.38
$N_{\delta_r}^\phi$	$A_\phi$	1.84	10.9	12.8	1.45	6.89	10.8	7.03	2.75	6.21
	$1/T_{\phi_1}$	2.78	5.37	8.29	2.48	4.35	6.64	5.53	3.27	4.43
	$1/T_{\phi_2}$	-4.11	-5.53	-8.18	-3.48	-4.68	-6.57	-4.76	-3.79	-4.61
$N_{\delta_r}^r$	$A_r$	-1.93	-8.61	-11.1	-1.56	-5.54	-8.80	-4.83	-2.54	-5.11
	$1/T_{r_1}$	1.13	4.33	9.87	0.553	2.35	6.12	7.31	0.578	2.64
	$\zeta_r$	0.538	0.475	0.674	0.414	0.473	0.535	0.790	0.440	0.526
	$\omega_r$	1.02	0.642	0.502	1.17	0.735	0.541	0.381	1.12	0.585
$N_{\delta_r}^\beta$	$A_\beta$	0.0430	0.0769	0.0626	0.0307	0.0537	0.0550	0.0192	0.0267	0.0347
	$1/T_{\beta_1}$	-0.0624	-0.00199	0.000266	-0.0603	-0.00616	0.000578	0.00271	-0.0178	-0.00216
	$1/T_{\beta_2}$	1.73	4.45	9.76	1.14	2.70	6.17	7.11	1.32	2.94
	$1/T_{\beta_3}$	54.7	120	186	63.6	113	170	272	110	160
$N_{\delta_r}^y$ CG	$A_y$	12.0	51.5	62.9	9.74	34.1	52.3	22.4	15.6	30.4
	$1/T_{y_1}$	-0.123	-0.0145	-0.00502	-0.108	-0.0227	-0.00654	0.000648	-0.0436	-0.0107
	$1/T_{y_2}$	1.87	4.43	9.57	1.27	2.69	6.16	7.06	1.36	2.97
	$1/T_{y_3}$	-2.00	-4.97	-7.84	-1.96	-3.69	-5.78	-8.91	-2.28	-3.81
	$1/T_{y_4}$	2.80	5.92	9.57	2.45	4.30	6.80	10.5	2.61	4.30

TABLE 4.6  
TYPICAL VALUES FOR LONGITUDINAL STABILITY DERIVATIVES

STABILITY DERIVATIVE	EQUATION	TYPICAL VALUE
$C_{x_u}$	$-2C_D - U_o [ \partial C_D / \partial u ]$	-0.05
$C_{x_\alpha}$	$C_L - [ \partial C_D / \partial \alpha ]$	+0.1
$C_{z_u}$	$-2C_L - U_o [ \partial C_L / \partial u ]$	-0.05
$C_{z_\alpha}$	$-C_D - [ \partial C_L / \partial \alpha ]$	-4
$C_{z_{\dot{\alpha}}}$		-1
$C_{z_q}$		-2
$C_{m_u}$	Neglect for Jets	
$C_{m_\alpha}$		-0.3
$C_{m_{\dot{\alpha}}}$		-3
$C_{m_q}$		-8

NOTE: EFFECTS OF  $C_{m_{\dot{\alpha}}}$  AND  $C_{m_q}$  ARE USUALLY COMBINED WHEN USING FLIGHT TEST DATA.

TABLE 4.7  
TYPICAL VALUES FOR LATERAL-DIRECTIONAL STABILITY DERIVATIVES

STABILITY DERIVATIVE	TYPICAL VALUE
$C_{y_\beta}$	-0.6
$C_{l_\beta}$	-0.06
$C_{l_p}$	-0.4
$C_{l_r}$	+0.06
$C_{n_\beta}$	+0.11
$C_{n_p}$	-0.015
$C_{n_r}$	-0.12
$C_{y_p}$	Neglect
$C_{y_r}$	Neglect

#### 4.13 LIST OF ABBREVIATIONS AND SYMBOLS

<u>Symbol</u>	<u>Definition</u>
A	Gain of root locus transfer function
b	Wingspan
c	Mean Aerodynamic Chord: The theoretical chord for a wing which has the same force vector as the actual wing (also MAC).
cg	Center of Gravity: common name for the aircraft's center of mass.
D	Drag: The component of the resultant aerodynamic force parallel to the relative wind. It must be specified whether this applies to a complete aircraft or to parts thereof.
$D_{wt}$	Drag due to weight.
$D_{0_{wt}}$	Initial drag due to weight.
f	function
$\bar{F}$	Applied force vector.
$F_x, F_y, F_z$	Components of applied forces on respective body axes.
$\bar{g}$	Local gravitational vector where g is the magnitude of the acceleration due to gravity.
$\bar{G}$	Applied moment vector.
$G_x, G_y, G_z$	Components of the applied moments on the respective body axes.
$\bar{H}$	Angular momentum vector.
$H_x, H_y, H_z$	Components of the angular momentum vector on the body axes.
$\bar{i}, \bar{j}, \bar{k}$	Unit vectors in the body axis system.
$I_x, I_y, I_z$	Moments of inertia about respective body axes. (Example: $I_x = \int_V \rho_A (y^2 + z^2) dV$ )
$I_{xy}, I_{yz}, I_{xz}$	Products of inertia, a measure of symmetry. (Example: $I_{xy} = I_{yx} = \int_V xy \rho_A dV$ )

<u>Symbol</u>	<u>Definition</u>
L	Lift: The component of the resultant aerodynamic force perpendicular to the relative wind. It must be specified whether this applies to a complete aircraft or to parts thereof.
$L_A, M_A, N_A$	Aerodynamic moments about the x, y and z vehicle axes.
$L_{oth}, M_{oth}, N_{oth}$	Other moments about the x, y and z vehicle axes.
$L_T, M_T, N_T$	Thrust moments about the x, y and z vehicle axes.
L, or l	Rolling moment.
$L_{wt}$	Lift due to weight
$L_{0_{wt}}$	Initial lift due to weight
LHS	Left hand side - the side of the equations of motion which represents the applied forces and moments on the aircraft.
M	Mach number
M, or m	Pitching moment
m	mass of aircraft
N, or n	Yawing Moment
P,Q,R	Angular rates about the x, y, and z vehicle axes.
$P_0, Q_0, R_0$	Initial angular rates.
p,q,r	Perturbed values of P,Q,R, respectively.
$\bar{r}$	Position vector measured from the cg.
Re	Reynolds number.
RPM	Revolutions per minute, used to indicate engine throttle setting.
RW	Relative wind
RHS	Right hand side - the side of the equations of motion which represents the aircraft's response to the applied forces and moments.
s	Laplace transform variable.
S	Wing area.
t	Time

<u>Symbol</u>	<u>Definition</u>
$T$	Thrust; temperature; time constant
$T_0$	Initial thrust.
$U, V, W$	Components of velocity along the x, y, and z vehicle axes.
$U_0, V_0, W_0$	Components of velocity along the x, y, and z vehicle axes at zero time (i.e., equilibrium condition).
$u, v, w$	Small perturbations of $U, V, W$ , respectively.
$\mathbf{X}$	Vector cross product.
$V$	Aircraft volume
$\bar{V}$	velocity
$\bar{V}_T$	True velocity (the velocity of the relative wind).
$wt$	Weight.
$X, Y, Z$	Axes in the earth axis system.
$\dot{X}, \dot{Y}, \dot{Z}$	Aircraft velocities in the inertial coordinate system.
$X_A, Y_A, Z_A$	Aerodynamic forces in the vehicle axes.
$X_g, Y_g, Z_g$	Gravity force in the vehicle axes.
$X_{g_0}, Y_{g_0}, Z_{g_0}$	Initial gravity forces.
$X_{oth}, Y_{oth}, Z_{oth}$	Other forces in the vehicle axes.
$X_T, Y_T, Z_T$	Thrust forces in the vehicle axes.
$x, y, z$	Axes in the body axis system.
$x_s, y_s, z_s$	Axes in the stability axis system.
$Y_{wt}$	Side force due to weight.
$Y_{0_{wt}}$	Initial side force due to weight.
$Z_k$	Distance between the thrust line and the cg.
$\rho$	Air density.
$\rho_A$	Mass density of aircraft.
$\alpha$	Angle of attack, also a small perturbation in $\alpha$ .

<u>Symbol</u>	<u>Definition</u>
$\alpha_0$	Initial $\alpha$ .
$\beta$	Sideslip angle, also a small perturbation in $\beta$ .
$\beta_0$	Initial $\beta$ .
$\Delta$	Characteristic equation; also used to denote a change in a variable.
$\delta_a, \delta_e, \delta_r$	Deflection angle of the ailerons, elevator, and rudder, respectively, also small perturbations in these values.
$\delta_{RPM}$	Change in engine throttle setting.
$\varepsilon$	Thrust angle.
$\sigma$	Total damping, $\sigma = \zeta\omega_n$ .
$\zeta$	Damping ratio.
$\psi, \theta, \phi$	Euler angles: yaw, pitch, and roll, respectively.
$\bar{\omega}$	Total angular velocity vector of an aircraft.
$\omega_d$	damped natural frequency.
$\omega_n$	natural frequency.
$\therefore$	therefore.

# PROBLEMS

- 4.1. Draw in the vectors  $\bar{V}_T$ ,  $\bar{U}$ ,  $\bar{V}$ ,  $\bar{W}$ , and show the angles  $\alpha$  and  $\beta$ .

Derive  $\alpha \approx W/V_T$  and  $\beta \approx V/V_T$  using the small angle assumption.

- 4.2. Define "right hand" and "orthogonal" with reference to a coordinate system.
- 4.3. Define "moving" earth and "fixed" earth axis coordinate systems.
- 4.4. Describe the "body" and "stability" axes systems.
- 4.5. Given  $\bar{F} = d/dt \ m\bar{V}_T$ ,  $\bar{F}$  is a force vector,  $m$  is a constant mass, and  $\bar{V}_T$  is the velocity vector of the mass center. Find  $F_x$ ,  $F_y$ , and  $F_z$  (if  $\bar{V}_T = U\bar{i} + V\bar{j} + W\bar{k}$  and  $\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k}$ ) with respect to the fixed earth axis system.
- 4.6. Given  $\bar{H} = \int_V \rho_A (\bar{r} \times \bar{V}) dV$  where  $\rho_A dV$  is the mass of a particle, with  $\bar{r}$  as its radius vector from the cg, and  $\bar{V}$  as its velocity, with respect to the cg. Find  $H_x$  with respect to the fixed earth axis system.
- 4.7. Write  $H_x$  in terms of  $I_x$ ,  $I_{xy}$ , and  $I_{xz}$ , given:

$$I_x = \int_V \rho_A (y^2 + z^2) dV$$

$$I_{xy} = \int_V \rho_A (xy) dV$$

$$I_{xz} = \int_V \rho_A (xz) dV$$

4.8. Draw the three views of a symmetric aircraft and explain why  $I_{xy} = 0$ ,  $I_{yz} = 0$ , and  $I_{xz} \neq 0$ . What is the aircraft's plane of symmetry?

4.9. Using the results from Problems 4.7 and 4.8, simplify the  $H_x$  equation.

4.10. If  $\bar{G} = d\bar{H}/dt |_{xyz}$ , use the following:

$$H_x = (PI_x - RI_{xz})$$

$$H_y = QI_y$$

$$H_z = (RI_z - PI_{xz})$$

to derive  $G_x$ ,  $G_y$ , and  $G_z$ .

4.11 Define:

- a. L
- b. M
- c. N
- d. P
- e. Q
- f. R

4.12. Define  $\psi$ ,  $\phi$ ,  $\theta$ . What are they used for? In what sequence must they be used? Explain the difference between  $\psi$  and  $\beta$ .

4.13. What are the expressions for P, Q, R, in terms of Euler angles?

4.14. Make a chart that has 5 columns and 6 rows. The columns should contain the terms of the left hand side of the equations of motion. Also, name each equation (pitch, drag, etc.). Fill in all terms.

4.15. What is the difference between straight flight and steady straight flight?

4.16.  $D, L, M = f( \quad , \quad , \quad , \quad , \quad )$

$Y, L, N = f( \quad , \quad , \quad , \quad , \quad , \quad )$

4.17. Write  $L/mU_0$  in terms of the stability parameters. Define  $L_u, L_\alpha, L_{\dot{\alpha}}, L_q, L_{\delta_e}$ .

4.18. Repeat 4.17 above for the other 5 equations.

4.19. Given 
$$\frac{M}{I_y} = M_u u + M_\alpha \alpha + M_{\dot{\alpha}} \dot{\alpha} + M_{\delta_e} \delta_e + M_q q$$

Where

$$M_u = \frac{1}{I_y} \frac{\partial M}{\partial u}, \quad M_\alpha = \frac{1}{I_y} \frac{\partial M}{\partial \alpha}, \quad M_{\dot{\alpha}} = \frac{1}{I_y} \frac{\partial M}{\partial \dot{\alpha}}$$

$$M_{\delta_e} = \frac{1}{I_y} \frac{\partial M}{\partial \delta_e}, \quad M_q = \frac{1}{I_y} \frac{\partial M}{\partial q}$$

Find:  $C_{m_u}, C_{m_\alpha}, C_{m_{\dot{\alpha}}}, C_{m_{\delta_e}},$  and  $C_{m_q}$ ;

(Note: the compensating factor for  $u$  is  $1/U_0$ ,  $\alpha$  and  $\delta_e$ 's compensating factor is 1, and  $\dot{\alpha}$  and  $q$ 's compensating factor is  $c/2U_0$ ).

4.20. Repeat Problem 4.19 for

(a)  $\frac{L}{I_x}$

(b)  $\frac{N}{I_z}$

(Note: The compensating factor for  $\beta, \delta_e,$  and  $\delta_r$  is 1. The compensating factor for  $\dot{\beta}, p,$  and  $r$  is  $b/2U_0$ ).

4.21. Given the following representative wind tunnel data:

$D_\theta = 32.2$	$D_u = -0.00620$
$D_\alpha = 0.0000848$	$L_q = 0$
$L_u = -0.0000385$	$L_\alpha = -1.16$
$L_{\delta_e} = -0.157$	$M_q = -0.696$
$M_u = 0.00104$	$M_{\dot{\alpha}} = -0.000210$
$M_{\delta_e} = -18.9$	$L_{\ddot{\alpha}} = 0$
$M_\alpha = -0.0143$	$D_{\delta_e} = 0$

Find a)  $\Delta$

b)  $G_{\delta_e}^\theta (s)$

c)  $G_{\delta_e}^q (s)$

d)  $N_{\delta_e}^{\dot{\alpha}} (s)$